EXISTENCE AND UNIQUENESS OF LINKING SYSTEMS: CHERMAK'S PROOF VIA OBSTRUCTION THEORY

BOB OLIVER

Abstract. We present a version of a proof by Andy Chermak of the existence and uniqueness of centric linking systems associated to arbitrary saturated fusion systems. This proof differs from the one in [Ch] in that it is based on the computation of higher derived functors of certain inverse limits. This leads to a much shorter proof, but one which is aimed mostly at researchers familiar with homological algebra.

One of the central questions in the study of fusion systems is whether to each saturated fusion system one can associate a centric linking system, and if so, whether it is unique. This question was recently answered positively by Andy Chermak [Ch], using direct constructions. His proof is quite lengthy, although some of the structures developed there seem likely to be of independent interest.

There is also a well established obstruction theory for studying this problem, involving higher derived functors of certain inverse limits. This is analogous to the use of group cohomology as an "obstruction theory" for the existence and uniqueness of group extensions. By using this theory, Chermak's proof can be greatly shortened, in part because it allows us to focus on the essential parts of Chermak's constructions, and in part by using results which are already established. The purpose of this paper is to present this shorter version of Chermak's proof, a form which we hope will be more easily accessible to researchers with a background in topology or homological algebra.

A *saturated fusion system* over a finite p-group S is a category whose objects are the subgroups of S, and whose morphisms are certain monomorphisms between the subgroups. This concept is originally due to Puig (see [P2]), and one version of his definition is given in Section 1 (Definition 1.1). One motivating example is the fusion system of finite group G with $S \in \mathrm{Syl}_p(G)$: the category $\mathcal{F}_S(G)$ whose objects are the subgroups of S and whose morphisms are those group homomorphisms which are conjugation by elements of G.

For $S \in \mathrm{Syl}_p(G)$ as above, there is a second, closely related category which can be defined, and which supplies the "link" between $\mathcal{F}_S(G)$ and the classifying space BG of G. A subgroup $P \le S$ is called *p-centric in G* if $Z(P) \in \mathrm{Syl}_p(C_G(P))$; equivalently, if $C_G(P) = Z(P) \times C'_G(P)$ for some (unique) subgroup $C'_G(P)$ of order

2000 *Mathematics Subject Classification.* Primary 55R35. Secondary 20J05, 20N99, 20D20, 20D05.

Key words and phrases. Finite groups, fusion, fusion systems, derived functors, classifying spaces, FF-offenders.

B. Oliver was partially supported by the DNRF through a visiting professorship at the Centre for Symmetry and Deformation in Copenhagen; and also by UMR 7539 of the CNRS and by project ANR BLAN08-2_338236, HGRT.

prime to p. Let $\mathcal{L}_S^c(G)$ (the *centric linking system of* G) be the category whose objects are the subgroups of S which are p-centric in G, and where for each pair of objects P, Q:

$$\mathrm{Mor}_{\mathcal{L}_S^c(G)}(P,Q) = \bigl\{ g \in G \bigm| {}^gP \le Q \bigr\} / C'_G(P) \ .$$

Such categories were originally defined by Puig in [P1].

To explain the significance of linking systems from a topologist's point of view, we must first define the *geometric realization* of an arbitrary small category $\mathcal{C}$. This is a space $|\mathcal{C}|$ built up of one vertex (point) for each object in $\mathcal{C}$, one edge for each nonidentity morphism (with endpoints attached to the vertices corresponding to its source and target), one 2-simplex (triangle) for each commutative triangle in $\mathcal{L}$, etc. (See, e.g., [AKO, § III.2.1–2] for more details.) By a theorem of Broto, Levi, and Oliver [BLO1, Proposition 1.1], for any G and S as above, the space $|\mathcal{L}_S^c(G)|$, after p-completion in the sense of Bousfield and Kan, is homotopy equivalent to the p-completed classifying space $BG^\wedge_p$ of G. Furthermore, many of the homotopy theoretic properties of the space $BG^\wedge_p$, such as its self homotopy equivalences, can be determined combinatorially by the properties (such as automorphisms) of the finite category $\mathcal{L}_S^c(G)$ [BLO1, Theorems B & C].

Abstract *centric linking systems* associated to a fusion system were defined in [BLO2] (see Definition 1.2). One of the motivations in [BLO2] for defining these categories was that it provides a way to associate a classifying space to a saturated fusion system. More precisely, if $\mathcal{L}$ is a centric linking system associated to a saturated fusion system $\mathcal{F}$, then we regard the p-completion $|\mathcal{L}|^\wedge_p$ of its geometric realization as a classifying space for $\mathcal{F}$. This is motivated by the equivalence $|\mathcal{L}_S^c(G)|^\wedge_p \simeq BG^\wedge_p$ noted above. To give one example of the role played by these classifying spaces, if $\mathcal{L}'$ is another centric linking system, associated to a fusion system $\mathcal{F}'$, and the classifying spaces $|\mathcal{L}|^\wedge_p$ and $|\mathcal{L}'|^\wedge_p$ are homotopy equivalent, then $\mathcal{L} \cong \mathcal{L}'$ and $\mathcal{F} \cong \mathcal{F}'$. We refer to [BLO2, Theorem A] for more details and discussion.

It is unclear from the definition whether there is a centric linking system associated to any given saturated fusion system, and if so, whether it is unique. Even when working with fusion systems of finite groups, which always have a canonical associated linking system, there is no simple reason why two groups with isomorphic fusion systems need have isomorphic linking systems, and hence equivalent p-completed classifying spaces. This question — whether $\mathcal{F}_S(G) \cong \mathcal{F}_T(H)$ implies $\mathcal{L}_S^c(G) \cong \mathcal{L}_T^c(H)$ and hence $BG^\wedge_p \simeq BH^\wedge_p$ — was originally posed by Martino and Priddy, and was what first got this author interested in the subject.

The main theorem of Chermak described in this paper is the following.

Theorem A (Chermak [Ch])**.** *Each saturated fusion system has an associated centric linking system, which is unique up to isomorphism.*

Proof. This follows immediately from Theorem 3.4 in this paper, together with [BLO2, Proposition 3.1]. □

In particular, this provides a new proof of the Martino-Priddy conjecture, which was originally proven in [O1, O2] using the classification of finite simple groups. Chermak's theorem is much more general, but it also (indirectly) uses the classification in its proof.

Theorem A is proven by Chermak by directly and systematically constructing the linking system, and by directly constructing an isomorphism between two given linking systems. The proof given here follows the same basic outline, but uses as its main tool the obstruction theory which had been developed in [BLO2, Proposition 3.1] for dealing with this problem. So if this approach is shorter, it is only because we are able to profit by the results of [BLO2, § 3], and also by other techniques which have been developed more recently for computing these obstruction groups.

By [BLO3, Proposition 4.6], there is a bijective correspondence between centric linking systems associated to a given saturated fusion system $\mathcal{F}$ up to isomorphism, and homotopy classes of rigidifications of the homotopy functor $\mathcal{O}(\mathcal{F}^c) \longrightarrow \mathsf{hoTop}$ which sends P to BP. (See Definition 1.4 for the definition of $\mathcal{O}(\mathcal{F}^c)$.) Furthermore, if $\mathcal{L}$ corresponds to a rigidification $\widetilde{B}$, then $|\mathcal{L}|$ is homotopy equivalent to the homotopy direct limit of $\widetilde{B}$. Thus another consequence of Theorem A is:

Theorem B. *For each saturated fusion system $\mathcal{F}$, there is a functor*

$$\widetilde{B}\colon \mathcal{O}(\mathcal{F}^c) \longrightarrow \mathsf{Top},$$

together with a choice of homotopy equivalences $\widetilde{B}(P) \simeq BP$ for each object P, such that for each $[\varphi] \in \mathrm{Mor}_{\mathcal{O}(\mathcal{F}^c)}(P,Q)$, the composite

$$BP \simeq \widetilde{B}(P) \xrightarrow{\widetilde{B}([\varphi])} \widetilde{B}(Q) \simeq BQ$$

is homotopic to $B\varphi$. Furthermore, $\widetilde{B}$ is unique up to homotopy equivalence of functors, and $\mathrm{hocolim}(\widetilde{B})^\wedge_p$ *is the (unique) classifying space for $\mathcal{F}$.*

We also want to compare "outer automorphism groups" of fusion systems, linking systems, and their classifying spaces. When $\mathcal{F}$ is a saturated fusion system over a p-group S, set

$$\mathrm{Aut}(S,\mathcal{F}) = \{\alpha \in \mathrm{Aut}(S) \,|\, {}^\alpha\mathcal{F} = \mathcal{F}\} \quad \text{and} \quad \mathrm{Out}(S,\mathcal{F}) = \mathrm{Aut}(S,\mathcal{F})/\mathrm{Aut}_{\mathcal{F}}(S)\ .$$

Here, for $\alpha \in \mathrm{Aut}(S)$, ${}^\alpha\mathcal{F}$ is the fusion system over S for which $\mathrm{Hom}_{{}^\alpha\mathcal{F}}(P,Q) = \alpha\circ\mathrm{Hom}_{\mathcal{F}}(\alpha^{-1}(P),\alpha^{-1}(Q))\circ\alpha^{-1}$. Thus $\mathrm{Aut}(S,\mathcal{F})$ is the group of "fusion preserving" automorphisms of S.

When $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$, then for each object P of $\mathcal{L}$, there is a "distinguished monomorphism" $\delta_P\colon P \longrightarrow \mathrm{Aut}_{\mathcal{L}}(P)$ (Definition 1.2). An automorphism α of $\mathcal{L}$ (a bijective functor from $\mathcal{L}$ to itself) is called *isotypical* if it permutes the images of the distinguished monomorphisms; i.e., if $\alpha(\delta_P(P)) = \delta_{\alpha(P)}(\alpha(P))$ for each P. We denote by $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ the group of isotypical automorphisms of $\mathcal{L}$ modulo natural transformations of functors. See also [AOV, § 2.2] or [AKO, Lemma III.4.9] for an alternative description of this group.

By [BLO2, Theorem D], $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) \cong \mathrm{Out}(|\mathcal{L}|^\wedge_p)$, where $\mathrm{Out}(|\mathcal{L}|^\wedge_p)$ is the group of homotopy classes of self homotopy equivalences of the space $|\mathcal{L}|^\wedge_p$. This is one reason for the importance of this particular group of (outer) automorphisms of $\mathcal{L}$. Another reason is the role played by $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ in the definition of a *tame fusion system* in [AOV, § 2.2].

The other main consequence of the results in this paper is the following.

Theorem C. *For each saturated fusion system $\mathcal{F}$ over a p-group S with associated centric linking system $\mathcal{L}$, the natural homomorphism*

$$\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) \xrightarrow{\ \mu_{\mathcal{L}}\ } \mathrm{Out}(S, \mathcal{F})$$

induced by restriction to $\delta_S(S) \cong S$ is surjective, and is an isomorphism if p is odd.

Proof. By [AKO, III.5.12], $\mathrm{Ker}(\mu_{\mathcal{L}}) \cong \varprojlim^1(\mathcal{Z}_{\mathcal{F}})$, and $\mu_{\mathcal{L}}$ is onto whenever $\varprojlim^2(\mathcal{Z}_{\mathcal{F}}) = 0$. (This was shown in [BLO1, Theorem E] when $\mathcal{L}$ is the linking system of a finite group.) So the result follows from Theorem 3.4 in this paper. □

I would like to thank Assaf Libman for very carefully reading this manuscript, pointing out a couple gaps in the arguments, and making other suggestions for improving it. And, of course, I very much want to thank Andy Chermak for solving this problem, which has taken up so much of my time for the past ten years.

1. Notation and background

We first briefly recall the definitions of saturated fusion systems and centric linking systems. For any group G and any pair of subgroups $H, K \leq G$, set

$$\mathrm{Hom}_G(H,K) = \left\{ c_g = (x \mapsto gxg^{-1}) \,\middle|\, g \in G,\ {}^gH \leq K \right\} \subseteq \mathrm{Hom}(H,K).$$

A *fusion system* $\mathcal{F}$ over a finite p-group S is a category whose objects are the subgroups of S, and whose morphism sets $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ satisfy the following two conditions:

- $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$ for all $P, Q \leq S$.
- For each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, $\varphi^{-1} \in \mathrm{Hom}_{\mathcal{F}}(\varphi(P), P)$.

Two subgroups $P, P' \leq S$ are called *$\mathcal{F}$-conjugate* if they are isomorphic in the category $\mathcal{F}$. Let $P^{\mathcal{F}}$ denote the set of subgroups $\mathcal{F}$-conjugate to P.

The following is the definition of a saturated fusion system first formulated in [BLO2]. Other (equivalent) definitions, including the original one by Puig, are discussed and compared in [AKO, §§ I.2 & I.9].

Definition 1.1. Let $\mathcal{F}$ be a fusion system over a p-group S.

- A subgroup $P \leq S$ is *fully centralized in* $\mathcal{F}$ if $|C_S(P)| \geq |C_S(Q)|$ for all $Q \in P^{\mathcal{F}}$.
- A subgroup $P \leq S$ is *fully normalized in* $\mathcal{F}$ if $|N_S(P)| \geq |N_S(Q)|$ for all $Q \in P^{\mathcal{F}}$.
- A subgroup $P \leq S$ is *$\mathcal{F}$-centric* if $C_S(Q) \leq Q$ for all $Q \in P^{\mathcal{F}}$.
- The fusion system $\mathcal{F}$ is *saturated* if the following two conditions hold:
  - (I) For all $P \leq S$ which is fully normalized in $\mathcal{F}$, P is fully centralized in $\mathcal{F}$ and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.
  - (II) If $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ are such that $\varphi(P)$ is fully centralized, and if we set

    $$N_\varphi = \{ g \in N_S(P) \,|\, \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi(P)) \},$$

    then there is $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\overline{\varphi}|_P = \varphi$.

For any fusion system $\mathcal{F}$ over S, let $\mathcal{F}^c \subseteq \mathcal{F}$ be the full subcategory whose objects are the $\mathcal{F}$-centric subgroups of S, and also let $\mathcal{F}^c$ denote the set of $\mathcal{F}$-centric subgroups of S.

Definition 1.2 ([BLO2])**.** Let $\mathcal{F}$ be a fusion system over the p-group S. A *centric linking system* associated to $\mathcal{F}$ is a category $\mathcal{L}$ with $\mathrm{Ob}(\mathcal{L}) = \mathcal{F}^c$, together with a functor $\pi\colon \mathcal{L} \longrightarrow \mathcal{F}^c$ and *distinguished monomorphisms* $P \xrightarrow{\delta_P} \mathrm{Aut}_{\mathcal{L}}(P)$ for each $P \in \mathrm{Ob}(\mathcal{L})$, which satisfy the following conditions.

(A) π is the identity on objects and is surjective on morphisms. For each $P, Q \in \mathcal{F}^c$, $\delta_P(Z(P))$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P,Q)$ by composition, and π induces a bijection
$$\mathrm{Mor}_{\mathcal{L}}(P,Q)/\delta_P(Z(P)) \xrightarrow{\;\cong\;} \mathrm{Hom}_{\mathcal{F}}(P,Q).$$

(B) For each $g \in P \in \mathcal{F}^c$, π sends $\delta_P(g) \in \mathrm{Aut}_{\mathcal{L}}(P)$ to $c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$.

(C) For each $P, Q \in \mathcal{F}^c$, $\psi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$, and $g \in P$, $\psi \circ \delta_P(g) = \delta_Q(\pi(\psi)(g)) \circ \psi$ in $\mathrm{Mor}_{\mathcal{L}}(P,Q)$.

We next fix some notation for sets of subgroups of a given group. For any group G, let $\mathscr{S}(G)$ be the set of subgroups of G. If $H \le G$ is any subgroup, set
$$\mathscr{S}(G)_{\ge H} = \{K \in \mathscr{S}(G) \mid K \ge H\}.$$

Definition 1.3. Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. An *interval* of subgroups of S is a subset $\mathcal{R} \subseteq \mathscr{S}(S)$ such that $P < Q < R$ and $P, R \in \mathcal{R}$ imply $Q \in \mathcal{R}$. An interval is $\mathcal{F}$*-invariant* if it is invariant under $\mathcal{F}$-conjugacy.

Thus, for example, an $\mathcal{F}$-invariant interval $\mathcal{R} \subseteq \mathscr{S}(S)$ is closed under overgroups if and only if $S \in \mathcal{R}$. Each $\mathcal{F}$-invariant interval has the form $\mathcal{R} \smallsetminus \mathcal{R}_0$ for some pair of $\mathcal{F}$-invariant intervals $\mathcal{R}_0 \subseteq \mathcal{R}$ which are closed under overgroups.

We next recall the obstruction theory to the existence and uniqueness of linking systems.

Definition 1.4. Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S.

(a) Let $\mathcal{O}(\mathcal{F}^c)$ be the *centric orbit category* of $\mathcal{F}$: $\mathrm{Ob}(\mathcal{O}(\mathcal{F}^c)) = \mathcal{F}^c$, and
$$\mathrm{Mor}_{\mathcal{O}(\mathcal{F}^c)}(P,Q) = \mathrm{Inn}(Q)\backslash \mathrm{Hom}_{\mathcal{F}}(P,Q).$$

(b) Let $\mathcal{Z}_{\mathcal{F}}\colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \longrightarrow \mathsf{Ab}$ be the functor which sends P to $Z(P) = C_S(P)$. If $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, and $[\varphi]$ denotes its class in $\mathrm{Mor}(\mathcal{O}(\mathcal{F}^c))$, then $\mathcal{Z}_{\mathcal{F}}([\varphi]) = \varphi^{-1}$ as a homomorphism from $Z(Q) = C_S(Q)$ to $Z(P) = C_S(P)$.

(c) For any $\mathcal{F}$-invariant interval $\mathcal{R} \subseteq \mathcal{F}^c$, let $\mathcal{Z}_{\mathcal{F}}^{\mathcal{R}}$ be the subquotient functor of $\mathcal{Z}_{\mathcal{F}}$ where $\mathcal{Z}_{\mathcal{F}}^{\mathcal{R}}(P) = Z(P)$ if $P \in \mathcal{R}$ and $\mathcal{Z}_{\mathcal{F}}^{\mathcal{R}}(P) = 0$ otherwise.

(d) For each $\mathcal{F}$-invariant interval $\mathcal{R} \subseteq \mathcal{F}^c$, we write for short
$$L^*(\mathcal{F};\mathcal{R}) = \varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^{*}(\mathcal{Z}_{\mathcal{F}}^{\mathcal{R}}) \;;$$
i.e., the higher derived functors of the inverse limit of $\mathcal{Z}_{\mathcal{F}}^{\mathcal{R}}$.

We refer to [AKO, § III.5.1] for more discussion of the functors $\varprojlim^*(-)$.

Thus $\mathcal{Z}_{\mathcal{F}} = \mathcal{Z}_{\mathcal{F}}^{\mathcal{F}^c}$, and $\varprojlim^*(\mathcal{Z}_{\mathcal{F}}) = L^*(\mathcal{F};\mathcal{F}^c)$. By [BLO2, Proposition 3.1], the obstruction to the existence of a centric linking system associated to $\mathcal{F}$ lies in $L^3(\mathcal{F};\mathcal{F}^c)$, and the obstruction to uniqueness lies in $L^2(\mathcal{F};\mathcal{F}^c)$.

For any $\mathcal{F}$ and any $\mathcal{F}$-invariant interval $\mathcal{R}$, $\mathcal{Z}_{\mathcal{F}}^{\mathcal{R}}$ is a quotient functor of $\mathcal{Z}_{\mathcal{F}}$ if $S \in \mathcal{R}$ (if $\mathcal{R}$ is closed under overgroups). If $\mathcal{R}_0 \subseteq \mathcal{R}$ are both $\mathcal{F}$-invariant intervals, and $P \in \mathcal{R}_0$ and $Q \in \mathcal{R} \smallsetminus \mathcal{R}_0$ implies $P \ngeq Q$, then $\mathcal{Z}_{\mathcal{F}}^{\mathcal{R}_0}$ is a subfunctor of $\mathcal{Z}_{\mathcal{F}}^{\mathcal{R}}$.

Lemma 1.5. *Fix a finite group Γ with Sylow subgroup $S \in \mathrm{Syl}_p(\Gamma)$, and set $\mathcal{F} = \mathcal{F}_S(\Gamma)$. Let $\mathcal{Q} \subseteq \mathcal{F}^c$ be an $\mathcal{F}$-invariant interval such that $S \in \mathcal{Q}$ (i.e., $\mathcal{Q}$ is closed under overgroups).*

(a) *Let $F\colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \longrightarrow \mathsf{Ab}$ be a functor such that $F(P) = 0$ for each $P \in \mathcal{F}^c \smallsetminus \mathcal{Q}$. Let $\mathcal{O}(\mathcal{F}_{\mathcal{Q}}) \subseteq \mathcal{O}(\mathcal{F}^c)$ be the full subcategory with object set $\mathcal{Q}$. Then*
$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^{*}(F) \cong \varprojlim_{\mathcal{O}(\mathcal{F}_{\mathcal{Q}})}{}^{*}(F|_{\mathcal{O}(\mathcal{F}_{\mathcal{Q}})}) \,.$$

(b) *Assume $\mathcal{Q} = \mathscr{S}(S)_{\geq Y}$ for some p-subgroup $Y \trianglelefteq \Gamma$ such that $C_\Gamma(Y) \leq Y$. Then*
$$L^k(\mathcal{F};\mathcal{Q}) \overset{\mathrm{def}}{=} \varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^{k}(\mathcal{Z}_{\mathcal{F}}^{\mathcal{Q}}) \cong \begin{cases} Z(\Gamma) & \text{if } k=0 \\ 0 & \text{if } k>0 \,. \end{cases}$$

Proof. **(a)** Set $\mathcal{C} = \mathcal{O}(\mathcal{F}^c)$ and $\mathcal{C}_0 = \mathcal{O}(\mathcal{F}_{\mathcal{Q}})$ for short. There is no morphism in $\mathcal{C}$ from any object of $\mathcal{C}_0$ to any object not in $\mathcal{C}_0$. Hence for any functor $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Ab}$ such that $F(P) = 0$ for each $P \notin \mathrm{Ob}(\mathcal{C}_0)$, the two chain complexes $C^*(\mathcal{C};F)$ and $C^*(\mathcal{C}_0;F|_{\mathcal{C}_0})$ are isomorphic (see, e.g., [AKO, § III.5.1]). So $\varprojlim^*(F) \cong \varprojlim^*(F|_{\mathcal{C}_0})$ in this situation, and this proves (a). Alternatively, (a) follows upon showing that any $\mathcal{C}_0$-injective resolution of $F|_{\mathcal{C}_0}$ can be extended to an $\mathcal{C}$-injective resolution of F by assigning to all functors the value zero on objects not in $\mathcal{C}_0$.

(b) To simplify notation, set $\overline{H} = H/Y$ for each $H \in \mathscr{S}(\Gamma)_{\geq Y}$, and $\bar{g} = gY \in \bar{\Gamma}$ for each $g \in \Gamma$. Let $\mathcal{O}_{\bar{S}}(\bar{\Gamma})$ be the "orbit category" of $\bar{\Gamma}$: the category whose objects are the subgroups of $\bar{S}$, and where for $P, Q \in \mathcal{Q}$,
$$\mathrm{Mor}_{\mathcal{O}_{\bar{S}}(\bar{\Gamma})}(\overline{P},\overline{Q}) = \overline{Q}\backslash\{g \in \bar{\Gamma} \,|\, {}^{g}\overline{P} \leq \overline{Q}\} \,.$$

There is an isomorphism of categories $\Psi\colon \mathcal{O}(\mathcal{F}_{\mathcal{Q}}) \xrightarrow{\ \cong\ } \mathcal{O}_{\bar{S}}(\bar{\Gamma})$ which sends $P \in \mathcal{Q}$ to $\overline{P} = P/Y$ and sends $[c_g] \in \mathrm{Mor}_{\mathcal{O}(\mathcal{F}_{\mathcal{Q}})}(P,Q)$ to $\overline{Q}\bar{g}$. Then $\mathcal{Z}_{\mathcal{F}}^{\mathcal{Q}} \circ \Psi^{-1}$ sends $\overline{P}$ to $Z(P) = C_{Z(Y)}(\overline{P})$. Hence for $k \geq 0$,
$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^{k}(\mathcal{Z}_{\mathcal{F}}^{\mathcal{Q}}) \cong \varprojlim_{\mathcal{O}(\mathcal{F}_{\mathcal{Q}})}{}^{k}(\mathcal{Z}_{\mathcal{F}}^{\mathcal{Q}}|_{\mathcal{O}(\mathcal{F}_{\mathcal{Q}})}) \cong \varprojlim_{\mathcal{O}_{\bar{S}}(\bar{\Gamma})}{}^{k}(\mathcal{Z}_{\mathcal{F}}^{\mathcal{Q}} \circ \Psi^{-1}) \cong \begin{cases} C_{Z(Y)}(\bar{\Gamma}) = Z(\Gamma) & \text{if } k=0 \\ 0 & \text{if } k>0 \end{cases}$$

where the first isomorphism holds by (a), and the last by a theorem of Jackowski and McClure [JM, Proposition 5.14]. We refer to [JMO, Proposition 5.2] for more details on the last isomorphism. □

More tools for working with these groups come from the long exact sequence of derived functors induced by a short exact sequence of functors.

Lemma 1.6. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Let $\mathcal{Q}$ and $\mathcal{R}$ be $\mathcal{F}$-invariant intervals such that*

(i) *$\mathcal{Q} \cap \mathcal{R} = \varnothing$,*

(ii) *$\mathcal{Q} \cup \mathcal{R}$ is an interval, and*

(iii) $Q \in \mathcal{Q}$, $R \in \mathcal{R}$ *implies* $Q \not\leq R$.

Then $\mathcal{Z}_{\mathcal{F}}^{\mathcal{R}}$ *is a subfunctor of* $\mathcal{Z}_{\mathcal{F}}^{\mathcal{Q}\cup\mathcal{R}}$, $\mathcal{Z}_{\mathcal{F}}^{\mathcal{Q}\cup\mathcal{R}}/\mathcal{Z}_{\mathcal{F}}^{\mathcal{R}} \cong \mathcal{Z}_{\mathcal{F}}^{\mathcal{Q}}$, *and there is a long exact sequence*

$$\begin{gathered} 0 \longrightarrow L^0(\mathcal{F};\mathcal{R}) \longrightarrow L^0(\mathcal{F};\mathcal{Q}\cup\mathcal{R}) \longrightarrow L^0(\mathcal{F};\mathcal{Q}) \longrightarrow \cdots \\ \longrightarrow L^{k-1}(\mathcal{F};\mathcal{Q}) \longrightarrow L^k(\mathcal{F};\mathcal{R}) \longrightarrow L^k(\mathcal{F};\mathcal{Q}\cup\mathcal{R}) \longrightarrow L^k(\mathcal{F};\mathcal{Q}) \longrightarrow \cdots . \end{gathered}$$

In particular, the following hold.

(a) *If* $L^k(\mathcal{F};\mathcal{R}) \cong L^k(\mathcal{F};\mathcal{Q}) = 0$ *for some* $k \geq 0$, *then* $L^k(\mathcal{F};\mathcal{Q}\cup\mathcal{R}) = 0$.

(b) *Assume* $\mathcal{F} = \mathcal{F}_S(\Gamma)$, *where* $S \in \mathrm{Syl}_p(\Gamma)$, *and there is a normal* p*-subgroup* $Y \trianglelefteq \Gamma$ *such that* $C_\Gamma(Y) \leq Y$ *and* $\mathcal{Q}\cup\mathcal{R} = \mathscr{S}(S)_{\geq Y}$. *Then for each* $k \geq 2$, $L^{k-1}(\mathcal{F};\mathcal{Q}) \cong L^k(\mathcal{F};\mathcal{R})$. *Also, there is a short exact sequence*

$$1 \longrightarrow C_{Z(Y)}(\Gamma) \longrightarrow C_{Z(Y)}(\Gamma^*) \longrightarrow L^1(\mathcal{F};\mathcal{R}) \longrightarrow 1,$$

where $\Gamma^* = \big\langle g \in \Gamma \,\big|\, {}^gP \in \mathcal{Q} \text{ for some } P \in \mathcal{Q}\big\rangle$.

Proof. Condition (iii) implies that $\mathcal{Z}_{\mathcal{F}}^{\mathcal{R}}$ is a subfunctor of $\mathcal{Z}_{\mathcal{F}}^{\mathcal{Q}\cup\mathcal{R}}$, and it is then immediate from the definitions (and (i) and (ii)) that $\mathcal{Z}_{\mathcal{F}}^{\mathcal{Q}\cup\mathcal{R}}/\mathcal{Z}_{\mathcal{F}}^{\mathcal{R}} \cong \mathcal{Z}_{\mathcal{F}}^{\mathcal{Q}}$. The long exact sequence is induced by this short exact sequence of functors and the snake lemma. Point (a) now follows immediately.

Under the hypotheses in (b), by Lemma 1.5(b), $L^k(\mathcal{F};\mathcal{Q}\cup\mathcal{R}) = 0$ for $k > 0$ and $L^0(\mathcal{F};\mathcal{Q}\cup\mathcal{R}) \cong Z(\Gamma) = C_{Z(Y)}(\Gamma)$. The first statement in (b) thus follows immediately from the long exact sequence, and the second since $L^0(\mathcal{F};\mathcal{Q}) \cong C_{Z(Y)}(\Gamma^*)$ (by definition of inverse limits). □

We next consider some tools for making computations in the groups $\varprojlim^*(-)$ for functors on orbit categories.

Definition 1.7. Fix a finite group G and a $\mathbb{Z}[G]$-module M. Let $\mathcal{O}_p(G)$ be the category whose objects are the p-subgroups of G, and where $\mathrm{Mor}_{\mathcal{O}_p(G)}(P,Q) = Q\backslash\{g \in G \,|\, {}^gP \leq Q\}$. Define a functor $F_M\colon \mathcal{O}_p(G)^{\mathrm{op}} \longrightarrow \mathsf{Ab}$ by setting

$$F_M(P) = \begin{cases} M & \text{if } P = 1 \\ 0 & \text{if } P \neq 1 \,. \end{cases}$$

Here, $F_M(1) = M$ has the given action of $\mathrm{Aut}_{\mathcal{O}_p(G)}(1) = G$. Set

$$\Lambda^*(G;M) = \varprojlim_{\mathcal{O}_p(G)}{}^*(F_M).$$

These groups $\Lambda^*(G;M)$ provide a means of computing higher limits of functors on orbit categories which vanish except on one conjugacy class.

Proposition 1.8 ([BLO2, Proposition 3.2])**.** *Let* $\mathcal{F}$ *be a saturated fusion system over a* p*-group* S. *Let*

$$F\colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$$

be any functor which vanishes except on the isomorphism class of some subgroup $Q \in \mathcal{F}^c$. *Then*

$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^*(F) \cong \Lambda^*(\mathrm{Out}_{\mathcal{F}}(Q);F(Q)).$$

Upon combining Proposition 1.8 with the exact sequences of Lemma 1.6, we get the following corollary.

Corollary 1.9. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S, and let $\mathcal{R} \subseteq \mathcal{F}^c$ be an $\mathcal{F}$-invariant interval. Assume, for some $k \geq 0$, that $\Lambda^k(\mathrm{Out}_{\mathcal{F}}(P); Z(P)) = 0$ for each $P \in \mathcal{R}$. Then $L^k(\mathcal{F}; \mathcal{R}) = 0$.*

What makes these groups $\Lambda^*(-;-)$ so useful is that they vanish in many cases, as described by the following proposition.

Proposition 1.10 ([JMO, Proposition 6.1(i,ii,iii,iv)]). *The following hold for each finite group G and each $\mathbb{Z}_{(p)}[G]$-module M.*

(a) *If $p\nmid|G|$, then* $\Lambda^i(G; M) = \begin{cases} M^G & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases}$

(b) *Let $H = C_G(M)$ be the kernel of the G-action on M. Then $\Lambda^*(G; M) \cong \Lambda^*(G/H; M)$ if $p\nmid|H|$, and $\Lambda^*(G; M) = 0$ if $p\big||H|$.*

(c) *If $O_p(G) \neq 1$, then $\Lambda^*(G; M) = 0$.*

(d) *If $M_0 \leq M$ is a $\mathbb{Z}_{(p)}[G]$-submodule, then there is an exact sequence*

$$0 \longrightarrow \Lambda^0(G; M_0) \longrightarrow \Lambda^0(G; M) \longrightarrow \Lambda^0(G; M/M_0) \longrightarrow \cdots$$
$$\cdots \longrightarrow \Lambda^{n-1}(G; M/M_0) \longrightarrow \Lambda^n(G; M_0) \longrightarrow \Lambda^n(G; M) \longrightarrow \cdots .$$

The next lemma allows us in certain cases to replace the orbit category for one fusion system by that for a smaller one. For any saturated fusion system $\mathcal{F}$ over S and any $Q \leq S$, the *normalizer fusion system* $N_{\mathcal{F}}(Q)$ is defined as a fusion system over $N_S(Q)$ (cf. [AKO, Definition I.5.3]). If Q is fully normalized, then $N_{\mathcal{F}}(Q)$ is always saturated (cf. [AKO, Theorem I.5.5]).

Lemma 1.11. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S, fix a subgroup $Q \in \mathcal{F}^c$ which is fully normalized in $\mathcal{F}$, and set $\mathcal{E} = N_{\mathcal{F}}(Q)$. Set $\mathcal{E}^\bullet = \mathcal{F}^c \cap \mathcal{E}^c$, a full subcategory of $\mathcal{E}^c$, and let $\mathcal{O}(\mathcal{E}^\bullet) \subseteq \mathcal{O}(\mathcal{E}^c)$ be its orbit category. Define*

$$\mathcal{T} = \bigl\{ P \leq S \bigm| Q \trianglelefteq P, \text{ and } R \in Q^{\mathcal{F}},\ R \trianglelefteq P \text{ implies } R = Q \bigr\}.$$

Let $F \colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathtt{mod}$ be any functor which vanishes except on subgroups $\mathcal{F}$-conjugate to subgroups in $\mathcal{T}$, set $F_0 = F|_{\mathcal{O}(\mathcal{E}^\bullet)}$, and let $F_1 \colon \mathcal{O}(\mathcal{E}^c)^{\mathrm{op}} \longrightarrow \mathtt{Ab}$ be such that $F_1|_{\mathcal{O}(\mathcal{E}^\bullet)} = F_0$ and $F_1(P) = 0$ for all $P \in \mathcal{E}^c \smallsetminus \mathcal{F}^c$. Then restriction to $\mathcal{E}^\bullet$ induces isomorphisms

$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^{*}(F) \xrightarrow[\cong]{\ R\ } \varprojlim_{\mathcal{O}(\mathcal{E}^\bullet)}{}^{*}(F_0) \xleftarrow[\cong]{\ R_1\ } \varprojlim_{\mathcal{O}(\mathcal{E}^c)}{}^{*}(F_1) \ . \tag{1}$$

Proof. Since R_1 is an isomorphism by Lemma 1.5(a), we only need to show that R is an isomorphism. If $F' \subseteq F$ is a pair of functors from $\mathcal{O}(\mathcal{F}^c)^{\mathrm{op}}$ to $\mathbb{Z}_{(p)}\text{-}\mathtt{mod}$, and the lemma holds for F' and for F/F', then it also holds for F by the 5-lemma (and since R and R_1 are both natural in F and preserve short exact sequences of functors). It thus suffices to prove that the maps in (1) are isomorphisms when F vanishes except on the $\mathcal{F}$-conjugacy class of one subgroup in $\mathcal{T}$.

Fix $P \in \mathcal{T}$, and assume $F(R) = 0$ for all $R \notin P^{\mathcal{F}}$. Thus $Q \trianglelefteq P$ by definition of $\mathcal{T}$. If $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ is such that $Q \trianglelefteq \varphi(P)$, then $\varphi^{-1}(Q) \trianglelefteq P$, so $\varphi(Q) = Q$

($\varphi \in \mathrm{Aut}_{\mathcal{E}}(Q)$) since $P \in \mathcal{T}$. Thus $\mathrm{Out}_{\mathcal{E}}(P) = \mathrm{Out}_{\mathcal{F}}(P)$, and $Q \trianglelefteq P^* \in P^{\mathcal{F}}$ implies $P^* \in P^{\mathcal{E}}$. This yields the following diagram:

$$\begin{tikzcd}
\varprojlim_{\mathcal{O}(\mathcal{F}^c)}^*(F) \arrow[r, "R"] \arrow[dr, "\Phi^*", "\cong"'] & \varprojlim_{\mathcal{O}(\mathcal{E}^\bullet)}^*(F_0) & \varprojlim_{\mathcal{O}(\mathcal{E}^c)}^*(F_1) \arrow[l, "R_1"', "\cong"] \arrow[dl, "\Phi_1^*"', "\cong"] \\
& \Lambda^*(\mathrm{Out}_{\mathcal{F}}(P); F(P)), &
\end{tikzcd}$$

where Φ^* and Φ_1^* are the isomorphisms of Proposition 1.8. The commutativity of the diagram follows from the precise description of Φ^* and Φ_1^* in [BLO2, Proposition 3.2]. Thus R is an isomorphism. □

The following lemma can also be stated and proven as a result about extending automorphisms from a linking system to a group [Ch, Lemma 4.17].

Lemma 1.12 ([Ch, 4.17])**.** *Fix a pair of finite groups $H \trianglelefteq G$, together with $S \in \mathrm{Syl}_p(G)$ and $T = S \cap H \in \mathrm{Syl}_p(H)$. Set $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{E} = \mathcal{F}_T(H)$. Assume $Y \leq T$ is such that $Y \trianglelefteq G$ and $C_G(Y) \leq Y$. Let $\mathcal{Q}$ be an $\mathcal{F}$-invariant interval in $\mathscr{S}(S)_{\geq Y}$ such that $S \in \mathcal{Q}$, and such that $Q \in \mathcal{Q}$ implies $H \cap Q \in \mathcal{Q}$. Set $\mathcal{Q}_0 = \{Q \in \mathcal{Q} \,|\, Q \leq H\}$. Then restriction induces an injective homomorphism*

$$L^1(\mathcal{F}; \mathcal{Q}) \xrightarrow{\;R\;} L^1(\mathcal{E}; \mathcal{Q}_0).$$

Proof. Since $\mathcal{E}^c$ need not be contained in $\mathcal{F}^c$, we must first check that there is a well defined "restriction" homomorphism. Set $\mathcal{E}^\bullet = \mathcal{E}^c \cap \mathcal{F}^c$: a full subcategory of $\mathcal{E}^c$. Since the functor $\mathcal{Z}_{\mathcal{E}}^{\mathcal{Q}_0}$ vanishes on all subgroups in $\mathcal{E}^c$ not in $\mathcal{Q}_0 \subseteq \mathcal{E}^\bullet$, the higher limits are the same whether taken over $\mathcal{O}(\mathcal{E}^\bullet)$ or $\mathcal{O}(\mathcal{E}^c)$ (Lemma 1.5(a)). Thus R is defined as the restriction map to $\varprojlim^1(\mathcal{Z}_{\mathcal{E}}^{\mathcal{Q}_0}|_{\mathcal{E}^\bullet}) \cong L^1(\mathcal{E}; \mathcal{Q}_0)$.

We work with the bar resolutions for $\mathcal{O}(\mathcal{F}^c)$ and $\mathcal{O}(\mathcal{E}^\bullet)$, using the notation of [AKO, § III.5.1]. Fix a cocycle $\eta \in Z^1(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}_{\mathcal{F}}^{\mathcal{Q}})$ such that $[\eta] \in \mathrm{Ker}(R)$. Thus η is a function from $\mathrm{Mor}(\mathcal{O}(\mathcal{F}^c))$ to $Z(Y)$ which sends the class $[\varphi]$ of $\varphi \in \mathrm{Hom}_G(P, Q)$ to an element of $Z(P)$ if $P \in \mathcal{Q}$, and to 1 if $P \notin \mathcal{Q}$. We can assume, after adding an appropriate coboundary, that $\eta\big(\mathrm{Mor}(\mathcal{O}(\mathcal{E}^\bullet))\big) = 1$.

Define $\widehat{\eta} \in Z^1(N_G(T)/T; Z(T))$ to be the restriction of η to $\mathrm{Aut}_{\mathcal{O}(\mathcal{F}^c)}(T) = N_G(T)/T$. For $g \in N_G(T)$, let $\bar{g}$ be its class in $N_G(T)/T$. Set $\gamma = \eta([\mathrm{incl}_T^S]) \in Z(T)$, so $d\gamma \in Z^1(N_G(T)/T; Z(T))$ is the cocycle $d\gamma(\bar{g}) = {}^{g}\gamma{\cdot}\gamma^{-1}$. For each $g \in S$, $[\mathrm{incl}_T^S]{\circ}[c_g] = [\mathrm{incl}_T^S]$ in $\mathcal{O}(\mathcal{F}^c)$, so ${}^{g}\gamma{\cdot}\eta([c_g]) = \gamma$, and thus $\widehat{\eta}(\bar{g}) = \eta([c_g]) = (d\gamma(\bar{g}))^{-1}$. In other words, $\widehat{\eta}|_{S/T}$ is a coboundary, and since $S/T \in \mathrm{Syl}_p(N_G(T)/T)$, $[\widehat{\eta}] = 1 \in H^1(N_G(T)/T; Z(T))$ (cf. [CE, Theorem XII.10.1]). Hence there is $\beta \in Z(T)$ such that $\widehat{\eta} = d\beta$. Since $\eta([c_h]) = 1$ for all $h \in N_H(T)$, $[\beta, h] = 1$ for all $h \in N_H(T)$, and thus $\beta \in Z(N_H(T))$.

Let $G^* \leq G$ be the subgroup generated by all $g \in G$ such that for some $Q \in \mathcal{Q}$, ${}^{g}Q \in \mathcal{Q}$. Define $H^* \leq H$ similarly. Since $S \leq N_G(T) \leq G^*$ and $N_H(T) \leq H^*$, $S \in \mathrm{Syl}_p(G^*)$, $T \in \mathrm{Syl}_p(H^*)$, and $HG^* \geq HN_G(T) = G$ by the Frattini argument (Lemma 1.13(b)). If $g = ha$ where $h \in H$, $a \in N_G(T)$, and ${}^{g}Q \in \mathcal{Q}$ for some $Q \in \mathcal{Q}$, then ${}^{a}(Q \cap H)$ and ${}^{g}(Q \cap H) = {}^{g}Q \cap H$ are both in $\mathcal{Q}_0$, and thus $h \in H^*$. Since $N_G(T)$ normalizes H^*, this shows that $G^* = H^*N_G(T)$. So $G^* \cap H = (H^*N_G(T)) \cap H = H^*N_H(T) = H^*$. In particular, $H^* \trianglelefteq G^*$ and $G^*/H^* \cong G/H$.

For each $\varphi \in \mathrm{Hom}_H(P,Q)$ (where $Y \le P, Q \le T$), and each $g \in N_G(T)$, set ${}^g\varphi = c_g \varphi c_g^{-1} \in \mathrm{Hom}_H({}^gP, {}^gQ)$. Since $\eta([\varphi]) = \eta([{}^g\varphi]) = 1$, we have $\varphi^{-1}(\widehat{\eta}(\bar{g})) = \widehat{\eta}(\bar{g})$. Thus for each $g \in N_G(T)$, $\widehat{\eta}(\bar{g}) = \beta^g\beta^{-1}$ is invariant under the action of H^*; i.e., $\beta^g\beta^{-1} \in Z(H^*)$. So the class $[\beta] \in Z(N_H(T))/Z(H^*)$ is fixed under the action of $N_G(T)$ on this quotient.

Since $p\nmid[H^*{:}N_H(T)]$, and since $N_G(T)$ normalizes H^* and $N_H(T)$, the inclusion of $Z(H^*) = C_{Z(Y)}(H^*)$ into $Z(N_H(T)) = C_{Z(Y)}(N_H(T))$ is $N_G(T)$-equivariantly split by the trace homomorphism for the actions of $H^* \ge N_H(T)$ on $Z(Y)$. So the fixed subgroup for the $N_G(T)$-action on the quotient group $Z(N_H(T))/Z(H^*)$ is $Z(N_G(T))/Z(G^*)$. Thus $\beta \in Z(N_G(T))Z(H^*)$, and we can assume $\beta \in Z(H^*)$ without changing $d\beta = \widehat{\eta}$.

Define a 0-cochain $\widehat{\beta} \in C^0(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}_{\mathcal{F}}^{\mathcal{Q}})$ by setting $\widehat{\beta}(P) = \beta$ if $P \in \mathcal{Q}_0$ and $\widehat{\beta}(P) = 1$ otherwise. Then $\eta([\varphi]) = d\widehat{\beta}([\varphi])$ for all $\varphi \in \mathrm{Mor}(\mathcal{E}^\bullet)$ (since both vanish) and also for all $\varphi \in \mathrm{Aut}_G(T)$. Since $G = HN_G(T)$, each morphism in $\mathcal{F}$ between subgroups of T is the composite of a morphism in $\mathcal{E}$ and the restriction of a morphism in $\mathrm{Aut}_{\mathcal{F}}(T)$. Hence $\eta([\varphi]) = d\widehat{\beta}([\varphi])$ for all such morphisms φ (since η and $d\widehat{\beta}$ are both cocycles). Upon replacing η by $\eta(d\widehat{\beta})^{-1}$, we can assume η vanishes on all morphisms in $\mathcal{F}$ between subgroups of T.

For each $P \in \mathcal{Q}$, set $P_0 = P \cap T$ and let $i_P \in \mathrm{Hom}_G(P_0, P)$ be the inclusion. Then $\eta([i_P]) \in Z(P_0)$ (and $\eta([i_P]) = 1$ if $P \notin \mathcal{Q}$). For each $g \in P$, the relation $[i_P] = [i_P] \circ [c_g]$ in $\mathcal{O}(\mathcal{F}^c)$ (where $[c_g] \in \mathrm{Aut}_{\mathcal{O}(\mathcal{F}^c)}(P_0)$) implies that $\eta([i_P])$ is c_g-invariant. Thus $\eta([i_P]) \in Z(P)$. Let $\rho \in C^0(\mathcal{O}(\mathcal{F}^c); \mathcal{Z}_{\mathcal{F}}^{\mathcal{Q}})$ be the 0-cochain $\rho(P) = \eta([i_P])$ when $P \in \mathcal{Q}$ and $\rho(P) = 1$ if $P \in \mathcal{F}^c \smallsetminus \mathcal{Q}$. Thus $\rho(P) = 1$ if $P \le T$ by the initial assumptions on η. Then $d\rho([i_P]) = \eta([i_P])$ for each P, and $d\rho(\varphi) = 1 = \eta(\varphi)$ for each φ between subgroups of T. For each $\varphi \in \mathrm{Hom}_G(P,Q)$ in $\mathcal{F}^c$, let $\varphi_0 \in \mathrm{Hom}_G(P_0, Q_0)$ be its restriction; the relation $[\varphi] \circ [i_Q] = [i_P] \circ [\varphi_0]$ in $\mathcal{O}(\mathcal{F}^c)$ implies that $\eta([\varphi]) = d\rho([\varphi])$ since this holds for $[\varphi_0]$ and the inclusions. Thus $\eta = d\rho$, and so $[\eta] = 1$ in $L^1(\mathcal{F}; \mathcal{Q})$. □

We end the section by recalling a few elementary results about finite groups.

Lemma 1.13. (a) *If $Q > P$ are p-groups for some prime p, then $N_Q(P) > P$.*

(b) (Frattini argument) *If $H \trianglelefteq G$ are finite groups and $T \in \mathrm{Syl}_p(H)$, then $G = HN_G(T)$.*

Proof. See, for example, [Sz1, Theorems 2.1.6 & 2.2.7]. □

Lemma 1.14. *Let G be a finite group such that $O_p(G) = 1$, and assume G acts faithfully on an abelian p-group D. Then G acts faithfully on $\Omega_1(D)$.*

Proof. The subgroup $C_G(\Omega_1(D))$ is a normal p-subgroup of G (cf. [G, Theorem 5.2.4]), and hence is contained in $O_p(G) = 1$. □

2. The Thompson subgroup and offenders

The proof of the main theorem is centered around the Thompson subgroup of a p-group, and the FF-offenders for an action of a group on an abelian p-group. We first fix the terminology and notation which will be used.

Definition 2.1. (a) For any p-group S, set $d(S) = \sup\{|A| \bigm| A \le S \text{ abelian}\}$, let $\mathcal{A}(S)$ be the set of all abelian subgroups of S of order $d(S)$, and set $J(S) = \langle \mathcal{A}(S) \rangle$.

(b) Let G be a finite group which acts faithfully on the abelian p-group D. A *best offender* in G on D is an abelian subgroup $A \le G$ such that $|A||C_D(A)| \ge |B||C_D(B)|$ for each $B \le A$. (In particular, $|A||C_D(A)| \ge |D|$.) Let $\mathcal{A}_D(G)$ be the set of best offenders in G on D, and set $J_D(G) = \langle \mathcal{A}_D(G) \rangle$.

(c) Let Γ be a finite group, and let $D \trianglelefteq \Gamma$ be a normal abelian p-subgroup. Let $J(\Gamma, D) \le \Gamma$ be the subgroup such that $J(\Gamma, D)/C_\Gamma(D) = J_D(\Gamma/C_\Gamma(D))$.

Note, in the situation of point (c) above, that

$$D \le C_\Gamma(D) \le J(\Gamma, D) \le \Gamma \qquad \text{and} \qquad J(J(\Gamma, D), D) = J(\Gamma, D).$$

The relation between the Thompson subgroup $J(-)$ and best offenders is described by the next lemma and corollary.

Lemma 2.2. (a) *Assume G acts faithfully on a finite abelian p-group D. If A is a best offender in G on D, and U is an A-invariant subgroup of D, then $A/C_A(U)$ is a best offender in $N_G(U)/C_G(U)$ on U.*

(b) *Let S be a finite p-group, let $D \trianglelefteq S$ be a normal abelian subgroup, and set $G = S/C_S(D)$. Assume $A \in \mathcal{A}(S)$. Then the image of A in G is a best offender on D.*

Proof. We give here the standard proofs.

(a) Set $\overline{A} = A/C_A(U)$ for short. For each $\overline{B} = B/C_A(U) \le \overline{A}$,

$$|C_U(B)||C_D(A)| = |C_U(B)C_D(A)||C_U(B) \cap C_D(A)| \le |C_D(B)||C_U(A)|.$$

Also, $|B||C_D(B)| \le |A||C_D(A)|$ since A is a best offender on D, and hence

$$|\overline{B}||C_U(\overline{B})| = \frac{|B||C_U(B)|}{|C_A(U)|} \le \frac{|B||C_D(B)|}{|C_D(A)|} \cdot \frac{|C_U(A)|}{|C_A(U)|} \le |A| \cdot \frac{|C_U(A)|}{|C_A(U)|} = |\overline{A}||C_U(\overline{A})|.$$

Thus $\overline{A}$ is a best offender on U.

(b) Set $\overline{A} = A/C_A(D)$, identified with the image of A in G. Fix some $\overline{B} = B/C_A(D) \le \overline{A}$, and set $B^* = C_D(B)B$. This is an abelian group since D and B are abelian, and hence $|B^*| \le |A|$ since $A \in \mathcal{A}(S)$. Since $B \cap C_D(B) \le C_D(A)$,

$$|\overline{B}||C_D(\overline{B})| = \frac{|B||C_D(B)|}{|C_A(D)|} = \frac{|B^*||B \cap C_D(B)|}{|C_A(D)|} \le \frac{|A||C_D(A)|}{|C_A(D)|} = |\overline{A}||C_D(\overline{A})|.$$

Since this holds for all $\overline{B} \le \overline{A}$, $\overline{A}$ is a best offender on D. $\square$

The following corollary reinterprets Lemma 2.2 in terms of the groups $J(\Gamma, D)$ defined above.

Corollary 2.3. *Let Γ be a finite group, and let $D \trianglelefteq \Gamma$ be a normal abelian p-subgroup.*

(a) *If $U \le D$ is also normal in Γ, then $J(\Gamma, U) \ge J(\Gamma, D)$.*

(b) *If Γ is a p-group, then $J(\Gamma) \le J(\Gamma, D)$.*

An action of a group G on a group D is *quadratic* if $[G,[G,D]] = 1$. If D is abelian and G acts faithfully, then a *quadratic best offender* in G on D is an abelian subgroup $A \le G$ which is a best offender and whose action is quadratic.

Lemma 2.4. *Let G be a finite group which acts faithfully on an elementary abelian p-group V. If the action of G on V is quadratic, then G is also an elementary abelian p-group.*

Proof. We write V additively for convenience; thus $[g,v] = gv - v$ for $g \in G$ and $v \in V$. By an easy calculation, and since the action is quadratic, $[gh,v] = [g,v] + [h,v]$ for each $g,h \in G$ and $v \in V$. Thus $g \mapsto (v \mapsto [g,v])$ is a homomorphism from G to the additive group $\mathrm{End}(V)$, and is injective since the action is faithful. Since $\mathrm{End}(V)$ is an elementary abelian p-group, so is G. □

We will also need the following form of Timmesfeld's replacement theorem.

Theorem 2.5. *Let A and V be abelian p-groups. Assume A acts on V, and is a best offender on V. Then there is $1 \ne B \le A$ such that B is a quadratic best offender on V. More precisely, we can take $B = C_A([A,V]) \ne 1$, in which case $|A||C_V(A)| = |B||C_V(B)|$ and $C_V(B) = [A,V] + C_V(A) < V$.*

Proof. We follow the proof given by Chermak in [Ch1, § 1]. Set $m = |A||C_V(A)|$. Since A is a best offender,

$$m = \sup\bigl\{|B||C_U(B)| \bigm| B \le A,\ U \le V\bigr\}. \tag{1}$$

For each $U \le V$, consider the set

$$\mathcal{M}_U = \bigl\{B \le A \bigm| |B||C_U(B)| = m\bigr\}\ .$$

By the maximality of m in (1),

$$B \in \mathcal{M}_U \quad\Longrightarrow\quad |C_U(B)| = |C_V(B)| \quad\Longrightarrow\quad C_V(B) \le U\ . \tag{2}$$

Step 1: (Thompson's replacement theorem) For each $x \in V$, set

$$V_x = [A,x] \overset{\text{def}}{=} \bigl\langle [a,x] = ax - x \bigm| a \in A\bigr\rangle \qquad\text{and}\qquad A_x = C_A(V_x).$$

Note that V_x is A-invariant. We will show that

$$|A_x||C_V(A_x)| = |A||C_V(A)| = m \qquad\text{and}\qquad C_V(A_x) = V_x + C_V(A). \tag{3}$$

Define $\Phi\colon A \longrightarrow V_x$ (a map of sets) by setting $\Phi(a) = [a,x] = ax - x$ for each $a \in A$. We first claim that Φ induces an injective map of sets

$$\phi\colon A/A_x \longrightarrow V_x/C_{V_x}(A)$$

between these quotient groups. Since A is abelian, $[a,[b,x]] = abx - bx - ax + x = [b,[a,x]]$ for all $a,b \in A$. Hence for all $g,h \in A$,

$$\begin{aligned}\Phi(g) - \Phi(h) = gx - hx \in C_{V_x}(A) \iff& h([h^{-1}g,x]) \in C_{V_x}(A)\\ \iff& 1 = [A,[h^{-1}g,x]] = [h^{-1}g,[A,x]] = [h^{-1}g,V_x]\\ \iff& h^{-1}g \in C_A(V_x) = A_x\ .\end{aligned}$$

Thus ϕ is well defined and injective.

Now,

$$|V_x||C_V(A)| = |C_{V_x}(A)||V_x + C_V(A)| \le |C_{V_x}(A)||C_V(A_x)|, \tag{4}$$

since $V_x \leq C_V(A_x)$ by definition of A_x. Together with the injectivity of ϕ, this implies that
$$\frac{|A|}{|A_x|} \leq \frac{|V_x|}{|C_{V_x}(A)|} \leq \frac{|C_V(A_x)|}{|C_V(A)|},$$
and so $m = |A||C_V(A)| \leq |A_x||C_V(A_x)|$. The opposite inequality holds by (1), so $A_x \in \mathcal{M}_V$ and the inequality in (4) is an equality. Thus $|V_x + C_V(A)| = |C_V(A_x)|$, finishing the proof of (3).

Step 2: Assume, for some $U \leq V$, that $B_0, B_1 \in \mathcal{M}_U$. Then $m = |B_0||C_U(B_0)| \geq |B_0B_1||C_U(B_0B_1)|$ by (1), and hence
$$\frac{|B_1|}{|B_0 \cap B_1|} = \frac{|B_0B_1|}{|B_0|} \leq \frac{|C_U(B_0)|}{|C_U(B_0B_1)|} = \frac{|C_U(B_0) + C_U(B_1)|}{|C_U(B_1)|} \leq \frac{|C_U(B_0 \cap B_1)|}{|C_U(B_1)|}.$$
So $m = |B_1||C_U(B_1)| \leq |B_0 \cap B_1||C_U(B_0 \cap B_1)|$ with equality by (1) again, and we conclude that $B_0 \cap B_1 \in \mathcal{M}_U$.

Step 3: Set $B = C_A([A, V])$ and $U = [A, V] + C_V(A)$. For each $x \in V$, (3) implies that $C_V(A_x) = [A, x] + C_V(A) \leq U$ and $A_x \in \mathcal{M}_U$. Hence $B = \bigcap_{x \in V} A_x \in \mathcal{M}_U$ by Step 2, so B is a best offender on V, and is quadratic since $[B, [A, V]] = 1$ by definition. Also, $C_V(B) \leq U$ by (2). Since $U = [A, V] + C_V(A) \leq C_V(B)$ by definition, we conclude that $U = C_V(B)$.

If $U = V$, then $V = [A, V] \oplus W$ is an A-invariant splitting for some $W \leq C_V(A)$. But this would imply $[A, V] = [A, [A, V]] + [A, W] = [A, [A, V]]$, which is impossible since $[A, X] < X$ for any finite p-group X on which A acts. We conclude that $U = C_V(B) < V$, and hence that $B \neq 1$. □

3. Proof of the main theorem

The following terminology will be very useful when carrying out the reduction procedures used in this section.

Definition 3.1 ([Ch, 6.3])**.** A *general setup* is a triple (Γ, S, Y), where Γ is a finite group, $S \in \mathrm{Syl}_p(\Gamma)$, $Y \trianglelefteq \Gamma$ is a normal p-subgroup, and $C_\Gamma(Y) \leq Y$ (Y is centric in Γ). A *reduced setup* is a general setup (Γ, S, Y) such that $Y = O_p(\Gamma)$, $C_S(Z(Y)) = Y$, and $O_p(\Gamma/C_\Gamma(Z(Y))) = 1$.

The next proposition, which will be shown in Section 4, is the key technical result needed to prove the main theorem. Its proof uses the classification by Meierfrankenfeld and Stellmacher [MS] of FF-offenders, and through that depends on the classification of finite simple groups.

Proposition 3.2 (Compare [Ch, 6.11])**.** *Let (Γ, S, Y) be a reduced setup, set $D = Z(Y)$, and assume $\Gamma/C_\Gamma(D)$ is generated by quadratic best offenders on D. Set $\mathcal{F} = \mathcal{F}_S(\Gamma)$, and let $\mathcal{R} \subseteq \mathcal{F}^c$ be the set of all $R \geq Y$ such that $J(R, D) = Y$. Then $L^2(\mathcal{F}; \mathcal{R}) = 0$ if $p = 2$, and $L^1(\mathcal{F}; \mathcal{R}) = 0$ if p is odd.*

Since this distinction between the cases where $p = 2$ or where p is odd occurs throughout this section and the next, it will be convenient to define
$$k(p) = \begin{cases} 2 & \text{if } p = 2 \\ 1 & \text{if } p \text{ is an odd prime.} \end{cases}$$

Thus under the hypotheses of Proposition 3.2, we claim that $L^{k(p)}(\mathcal{F};\mathcal{R})=0$.

Proposition 3.2 seems very restricted in scope, but it can be generalized to the following situation.

Proposition 3.3 (Compare [Ch, 6.12])**.** *Let (Γ,S,Y) be a general setup, and assume Proposition 3.2 holds for all reduced setups (Γ^*,S^*,Y^*) with $|\Gamma^*|\le|\Gamma|$. Set $\mathcal{F}=\mathcal{F}_S(\Gamma)$ and $D=Z(Y)$. Let $\mathcal{R}\subseteq\mathscr{S}(S)_{\ge Y}$ be an $\mathcal{F}$-invariant interval such that for each $Q\in\mathscr{S}(S)_{\ge Y}$, $Q\in\mathcal{R}$ if and only if $J(Q,D)\in\mathcal{R}$. Then $L^k(\mathcal{F};\mathcal{R})=0$ for all $k\ge k(p)$.*

Proof. Assume the proposition is false. Let $(\Gamma,S,Y,\mathcal{R},k)$ be a counterexample for which the 4-tuple $(k,|\Gamma|,|\Gamma/Y|,|\mathcal{R}|)$ is the smallest possible under the lexicographical ordering.

We will show in Step 1 that $\mathcal{R}=\{P\le S\,|\,J(P,D)=Y\}$, in Step 2 that $k=k(p)$, in Step 3 that (Γ,S,Y) is a reduced setup, and in Step 4 that $\Gamma/C_\Gamma(D)$ is generated by quadratic best offenders on D. The result then follows from Proposition 3.2.

Step 1: Let $R_0\in\mathcal{R}$ be a minimal element of $\mathcal{R}$ which is fully normalized in $\mathcal{F}$. Since $J(R_0,D)\in\mathcal{R}$ by assumption (and $J(R_0,D)\le R_0$), $J(R_0,D)=R_0$. Let $\mathcal{R}_0$ be the set of all $R\in\mathcal{R}$ such that $J(R,D)$ is $\mathcal{F}$-conjugate to R_0, and set $\mathcal{Q}_0=\mathcal{R}\smallsetminus\mathcal{R}_0$. Then $\mathcal{R}_0$ and $\mathcal{Q}_0$ are both $\mathcal{F}$-invariant intervals, and satisfy the conditions $Q\in\mathcal{R}_0$ ($Q\in\mathcal{Q}_0$) if and only if $J(Q,D)\in\mathcal{R}_0$ ($J(Q,D)\in\mathcal{Q}_0$). Since $L^k(\mathcal{F};\mathcal{R})\ne0$, Lemma 1.6(a) implies $L^k(\mathcal{F};\mathcal{R}_0)\ne0$ or $L^k(\mathcal{F};\mathcal{Q}_0)\ne0$. Hence $\mathcal{Q}_0=\varnothing$ and $\mathcal{R}=\mathcal{R}_0$ by the minimality assumption on $|\mathcal{R}|$ (and since $\mathcal{R}_0\ne\varnothing$).

Set $\Gamma_1=N_\Gamma(R_0)$, $S_1=N_S(R_0)$, $\mathcal{F}_1=N_\mathcal{F}(R_0)=\mathcal{F}_{S_1}(\Gamma_1)$ (see [AKO, Proposition I.5.4]), and $\mathcal{R}_1=\{R\in\mathcal{R}\,|\,J(R,D)=R_0\}$. Every subgroup in $\mathcal{R}$ is $\mathcal{F}$-conjugate to a subgroup in $\mathcal{R}_1$. Also, for $P\in\mathcal{R}_1$, if $R\in R_0{}^{\mathcal{F}}$ and $R\trianglelefteq P$, then $J(P,D)\ge J(R,D)=R$ implies $R=R_0$. The hypotheses of Lemma 1.11 are thus satisfied, and so $L^k(\mathcal{F}_1;\mathcal{R}_1)\cong L^k(\mathcal{F};\mathcal{R})\ne0$. Thus $(\Gamma_1,S_1,Y,\mathcal{R}_1,k)$ is another a counterexample to the proposition. By the minimality assumption, $\Gamma_1=\Gamma$, and thus $R_0\trianglelefteq\Gamma$.

We have now shown that there is a p-subgroup $R_0\trianglelefteq\Gamma$ such that $\mathcal{R}=\{R\le S\,|\,J(R,D)=R_0\}$. Set $Y_1=R_0\ge Y$ and $D_1=Z(Y_1)\le D$. By Corollary 2.3(a), for each $R\le S$ such that $R\ge R_0$ and $R\notin\mathcal{R}$, $J(R,D_1)\ge J(R,D)\notin\mathcal{R}$, and hence $J(R,D_1)\notin\mathcal{R}$. Thus $(\Gamma,S,Y_1,\mathcal{R},k)$ is a counterexample to the proposition, and so $Y=Y_1=R_0$ by the minimality assumption on $|\Gamma/Y|$. We conclude that $\mathcal{R}=\{R\le S\,|\,J(R,D)=Y\}$.

Step 2: Let $\mathcal{Q}$ be the set of all overgroups of Y in S which are not in $\mathcal{R}$. Equivalently, $\mathcal{Q}=\{Q\le S\,|\,J(Q,D)>Y\}$. If $k\ge2$, then $L^{k-1}(\mathcal{F};\mathcal{Q})\cong L^k(\mathcal{F};\mathcal{R})\ne0$ by Lemma 1.6(b). Since k was assumed to be the smallest degree $\ge k(p)$ for which the proposition is not true, we conclude that $k=k(p)$.

Step 3: Assume (Γ,S,Y) is not a reduced setup. Let $K\trianglelefteq\Gamma$ be such that $K\ge C_\Gamma(D)$ and $K/C_\Gamma(D)=O_p(\Gamma/C_\Gamma(D))$, and set $Y_2=S\cap K\trianglelefteq S$. Then $Y_2>Y$, since either $Y_2\ge O_p(\Gamma)>Y$, or $Y_2\ge C_S(D)>Y$, or $p\big||K/C_\Gamma(D)|$ and hence $Y_2>C_S(D)\ge Y$. Set $\Gamma_2=N_\Gamma(Y_2)$, and set $\mathcal{R}_2=\{P\in\mathcal{R}\,|\,P\ge Y_2\}$. Note that $S\in\mathrm{Syl}_p(\Gamma_2)$, and also that $\mathcal{R}_2$ is an $\mathcal{F}$-invariant interval since Y_2 is strongly closed in S with respect to Γ. Set $\mathcal{F}_2=\mathcal{F}_S(\Gamma_2)=N_\mathcal{F}(Y_2)$ [AKO, Proposition I.5.4].

Assume $P\in\mathcal{R}\smallsetminus\mathcal{R}_2$. Then $P\not\ge Y_2$, so $PY_2>P$, and hence $N_{PY_2}(P)>P$ (Lemma 1.13(a)). Set $G=\mathrm{Out}_\Gamma(P)$ and $G_0=\mathrm{Out}_K(P)$. Then $G_0\trianglelefteq G$ since

$K \trianglelefteq \Gamma$, and $C_{G_0}(Z(P)) = \mathrm{Out}_{C_K(Z(P))}(P) \geq \mathrm{Out}_{C_\Gamma(D)}(P)$ since $K \geq C_\Gamma(D)$ and $Z(P) \leq Z(Y) = D$. Hence $G_0/C_{G_0}(Z(P))$ is a p-group since $K/C_\Gamma(D)$ is a p-group. For any $g \in N_{PY_2}(P)\smallsetminus P$, $\mathrm{Id} \neq [c_g] \in \mathrm{Out}_K(P) = G_0$ since $Y_2 \leq K$ (and since $C_\Gamma(P) \leq C_\Gamma(Y) \leq Y \leq P$). Thus $\mathrm{Out}_K(P) = G_0 \trianglelefteq G$ contains a nontrivial element of p-power order, and its action on $Z(P)$ factors through the p-group $G_0/C_{G_0}(Z(P))$. Proposition 1.10(b,c) now implies that $\Lambda^*(\mathrm{Out}_\Gamma(P); Z(P)) = 0$.

Since this holds for all $P \in \mathcal{R}\smallsetminus\mathcal{R}_2$, $L^*(\mathcal{F}; \mathcal{R}\smallsetminus\mathcal{R}_2) = 0$ by Corollary 1.9. Hence $L^*(\mathcal{F}; \mathcal{R}_2) \cong L^*(\mathcal{F}; \mathcal{R})$ by the exact sequence in Lemma 1.6. Also, the hypotheses of Lemma 1.11 hold for the functor $\mathcal{Z}_\mathcal{F}^{\mathcal{R}_2}$ on $\mathcal{O}(\mathcal{F}^c)$ (with $Q = Y_2$) since Y_2 is strongly closed. So $L^*(\mathcal{F}; \mathcal{R}_2) \cong L^*(\mathcal{F}_2; \mathcal{R}_2)$. Since $L^k(\mathcal{F}; \mathcal{R}) \neq 0$ by assumption, $L^k(\mathcal{F}_2; \mathcal{R}_2) \neq 0$.

Set $D_2 = Z(Y_2) \leq D$. For each $P \in \mathscr{S}(S)_{\geq Y_2}$,

$$P \geq J(P, D_2) \geq J(P, D) \geq C_P(D) \geq Y \tag{1}$$

by Corollary 2.3(a) and by definition of $J(P, -)$. We must show that $P \in \mathcal{R}_2$ if and only if $J(P, D_2) \in \mathcal{R}_2$. If $P \in \mathcal{R}_2 \subseteq \mathcal{R}$, then $J(P, D) \in \mathcal{R}$ by assumption, so $J(P, D_2) \in \mathcal{R}$ by (1) since $\mathcal{R}$ is an interval, and $J(P, D_2) \in \mathcal{R}_2$ since $J(P, D_2) \geq C_P(D_2) \geq Y_2$. If $P \notin \mathcal{R}_2$, then $P \notin \mathcal{R}$, so $J(P, D) \notin \mathcal{R}$, and $J(P, D_2) \notin \mathcal{R}$ (hence $J(P, D_2) \notin \mathcal{R}_2$) by (1) again and since $\mathcal{R}$ is an interval containing Y.

Thus $(\Gamma_2, S, Y_2, \mathcal{R}_2, k)$ is a counterexample to the proposition. So $\Gamma_2 = \Gamma$ and $Y_2 = Y$ by the minimality assumption, which contradicts the above claim that $Y_2 > Y$. We conclude that (Γ, S, Y) is a reduced setup.

Step 4: It remains to prove that $\Gamma/C_\Gamma(D)$ is generated by quadratic best offenders on D; the result then follows from Proposition 3.2.

Let $\Gamma_3 \trianglelefteq \Gamma$ be such that $\Gamma_3 \geq C_\Gamma(D)$ and $\Gamma_3/C_\Gamma(D)$ is generated by all quadratic best offenders on D. If $\Gamma_3 = \Gamma$ we are done, so assume $\Gamma_3 < \Gamma$. Set $S_3 = \Gamma_3 \cap S$ and $\mathcal{F}_3 = \mathcal{F}_{S_3}(\Gamma_3)$. Set

$$\mathcal{Q} = \mathscr{S}(S)_{\geq Y}\smallsetminus\mathcal{R}, \quad \mathcal{Q}_3 = \mathcal{Q} \cap \mathscr{S}(S_3)_{\geq Y}, \quad \text{and} \quad \mathcal{R}_3 = \mathcal{R} \cap \mathscr{S}(S_3)_{\geq Y}.$$

Since $L^k(\mathcal{F}; \mathcal{R}) \neq 0$, $\mathcal{R} \subsetneqq \mathscr{S}(S)_{\geq Y}$ by Lemma 1.5(b), and $\mathcal{Q} \neq \varnothing$. The proposition holds for $(\Gamma_3, S_3, Y, \mathcal{R}_3, k)$ by the minimality assumption, and thus $L^k(\mathcal{F}_3; \mathcal{R}_3) = 0$.

For $Q \in \mathcal{Q}$, $J(Q, D) > Y$, so Q/Y has nontrivial best offenders on D, hence has nontrivial quadratic best offenders on D by Theorem 2.5, and thus $J(Q\cap\Gamma_3, D) > Y$. So $Q \in \mathcal{Q}$ implies $Q \cap \Gamma_3 \in \mathcal{Q}_3$ by Step 1. In particular, $S_3 \in \mathcal{Q}_3$.

If $k = 2$ (i.e., if $p = 2$), then $L^1(\mathcal{F}; \mathcal{Q}) \cong L^2(\mathcal{F}; \mathcal{R}) \neq 0$ and $L^1(\mathcal{F}_3; \mathcal{Q}_3) \cong L^2(\mathcal{F}_3; \mathcal{R}_3) = 0$ by Lemma 1.6(b), which is impossible by Lemma 1.12.

If $k = 1$ (if p is odd), set

$$\Gamma^* = \big\langle g \in \Gamma \,\big|\, {}^gP \in \mathcal{Q} \text{ for some } P \in \mathcal{Q} \big\rangle \leq \Gamma$$
$$\Gamma_3^* = \big\langle g \in \Gamma_3 \,\big|\, {}^gP \in \mathcal{Q}_3 \text{ for some } P \in \mathcal{Q}_3 \big\rangle \leq \Gamma_3 \;.$$

Then $\Gamma_3^* \leq \Gamma^*$ since $\Gamma_3 \leq \Gamma$ and $\mathcal{Q}_3 \subseteq \mathcal{Q}$. By Lemma 1.6(b), there are exact sequences

$$\begin{array}{c} 1 \longrightarrow C_{Z(Y)}(\Gamma^*) \longrightarrow C_{Z(Y)}(\Gamma^*) \longrightarrow L^1(\mathcal{F}; \mathcal{Q}) \neq 1 \\ 1 \longrightarrow C_{Z(Y)}(\Gamma_3) \longrightarrow C_{Z(Y)}(\Gamma_3^*) \longrightarrow L^1(\mathcal{F}_3; \mathcal{Q}_3) = 1. \end{array} \tag{2}$$

Also, $\Gamma^*\Gamma_3 \geq N_\Gamma(S_3)\Gamma_3 = \Gamma$ since $S_3 \in \mathcal{Q}_3$, where the equality follows from the Frattini argument (Lemma 1.13(b)), so

$$C_{Z(Y)}(\Gamma) = C_{Z(Y)}(\Gamma^*\Gamma_3) = C_{Z(Y)}(\Gamma^*) \cap C_{Z(Y)}(\Gamma_3).$$

But this is impossible, since $C_{Z(Y)}(\Gamma) < C_{Z(Y)}(\Gamma^*) \leq C_{Z(Y)}(\Gamma_3^*) = C_{Z(Y)}(\Gamma_3)$ by the exactness in (2). $\square$

We now have the tools needed to prove the main vanishing result.

Theorem 3.4. *For each saturated fusion system $\mathcal{F}$ over a p-group S, $\varprojlim^k_{\mathcal{Q}(\mathcal{F}^c)}(\mathcal{Z}_\mathcal{F}) = 0$ for all $k \geq 2$, and for all $k \geq 1$ if p is odd.*

Proof. As in [Ch, § 6], we choose inductively subgroups $X_0, X_1, \ldots, X_N \in \mathcal{F}^c$ and $\mathcal{F}$-invariant intervals $\varnothing = \mathcal{Q}_{-1} \subseteq \mathcal{Q}_0 \subseteq \cdots \subseteq \mathcal{Q}_N = \mathcal{F}^c$ as follows. Assume $\mathcal{Q}_{n-1}$ has been defined ($n \geq 0$), and $\mathcal{Q}_{n-1} \subsetneqq \mathcal{F}^c$. Consider the sets of subgroups

$$\begin{aligned}
\mathcal{U}_1 &= \bigl\{P \in \mathcal{F}^c \smallsetminus \mathcal{Q}_{n-1} \bigm| d(P) \text{ maximal}\bigr\} \\
\mathcal{U}_2 &= \bigl\{P \in \mathcal{U}_1 \bigm| |J(P)| \text{ maximal}\bigr\} \\
\mathcal{U}_3 &= \bigl\{P \in \mathcal{U}_2 \bigm| J(P) \in \mathcal{F}^c\bigr\} \\
\mathcal{U}_4 &= \begin{cases} \bigl\{P \in \mathcal{U}_3 \bigm| |P| \text{ minimal}\bigr\} & \text{if } \mathcal{U}_3 \neq \varnothing \\ \bigl\{P \in \mathcal{U}_2 \bigm| |P| \text{ maximal}\bigr\} & \text{otherwise.} \end{cases}
\end{aligned}$$

(See Definition 2.1(a) for the definition of $d(P)$.) Let X_n be any subgroup in $\mathcal{U}_4$ which is fully normalized in $\mathcal{F}$. Since $\mathcal{U}_4$ is invariant under $\mathcal{F}$-conjugacy, this is always possible.

Let $\mathcal{Q}_n$ be the union of $\mathcal{Q}_{n-1}$ with the set of all overgroups of subgroups $\mathcal{F}$-conjugate to X_n. Set $\mathcal{R}_n = \mathcal{Q}_n \smallsetminus \mathcal{Q}_{n-1}$ for each $0 \leq n \leq N$. By definition of $\mathcal{U}_4$, $X_n = J(X_n)$ if $J(X_n) \in \mathcal{F}^c$, while $\mathcal{R}_n = {X_n}^\mathcal{F}$ if $J(X_n) \notin \mathcal{F}^c$. Note also that $X_0 = J(S)$ and $\mathcal{R}_0 = \mathcal{Q}_0 = \mathscr{S}(S)_{\geq J(S)}$.

We will show, for each n, that

$$L^k(\mathcal{F}; \mathcal{R}_n) = 0 \text{ for all } k \geq k(p). \tag{3}$$

Then by Lemma 1.6(a), for all $k \geq k(p)$, $L^k(\mathcal{F}; \mathcal{Q}_{n-1}) = 0$ implies $L^k(\mathcal{F}; \mathcal{Q}_n) = 0$. The theorem now follows by induction on n.

Case 1: Assume n is such that $J(X_n) \notin \mathcal{F}^c$, and hence that $\mathcal{R}_n = {X_n}^\mathcal{F}$. Since $J(X_n)$ is centric in X_n but not in S, $X_nC_S(J(X_n)) > X_n$. Hence $N_{X_nC_S(J(X_n))}(X_n) > X_n$ (Lemma 1.13(a)), so there is $g \in N_S(X_n) \smallsetminus X_n$ such that $[g, J(X_n)] = 1$. Then g acts trivially on $Z(X_n) \leq J(X_n)$, so the kernel of the $\mathrm{Out}_\mathcal{F}(X_n)$-action on $Z(X_n)$ has order a multiple of p, and $\Lambda^*(\mathrm{Out}_\mathcal{F}(X_n); Z(X_n)) = 0$ by Proposition 1.10(b). Hence (3) holds by Proposition 1.8.

Case 2: Assume n is such that $J(X_n) \in \mathcal{F}^c$, and hence $X_n = J(X_n)$ by definition of $\mathcal{U}_4$. By definition of $\mathcal{U}_1$ and $\mathcal{U}_2$, for each $P \geq X_n$ in $\mathcal{R}_n$, $d(P) = d(X_n)$ and $J(P) = X_n$. Hence

$$P \in \mathcal{R}_n \implies J(P) \text{ is the unique subgroup of } P \text{ } \mathcal{F}\text{-conjugate to } X_n. \tag{4}$$

Set $T = N_S(X_n)$ and $\mathcal{E} = N_\mathcal{F}(X_n)$. Then $\mathcal{E}$ is a saturated fusion system over T (cf. [AKO, Theorem I.5.5]), and contains X_n as normal centric subgroup. Hence

there is a *model* for $\mathcal{E}$ (cf. [AKO, Theorem III.5.10]): a finite group Γ such that $T \in \mathrm{Syl}_p(\Gamma)$, $X_n \trianglelefteq \Gamma$, $C_\Gamma(X_n) \le X_n$, and $\mathcal{F}_T(\Gamma) = \mathcal{E}$.

Let $\mathcal{R}$ be the set of all $P \in \mathcal{R}_n$ such that $P \ge X_n$; thus $\mathcal{R} = \mathcal{R}_n \cap \mathcal{E}$ by (4). Then (Γ, T, X_n) is a general setup, and $\mathcal{R}$ is an $\mathcal{E}$-invariant interval containing X_n. If $P \in \mathcal{R}$ and $Y \le P$ is $\mathcal{F}$-conjugate to X_n, then $Y = X_n$ by (4). The hypotheses of Lemma 1.11 thus hold, and hence

$$L^*(\mathcal{F}; \mathcal{R}_n) \cong L^*(\mathcal{E}; \mathcal{R}). \tag{5}$$

Set $D = Z(X_n)$. We claim that for each $P \in \mathscr{S}(T)_{\ge X_n}$,

$$P \in \mathcal{R} \quad \Longleftrightarrow \quad J(P, D) \in \mathcal{R}\ . \tag{6}$$

Fix such a P. By Corollary 2.3(b), $J(P, D) \ge J(P)$, and $X_n \ge J(X_n, D) \ge J(X_n) = X_n$. If $P \in \mathcal{R}$, then $J(P, D) \in \mathcal{R}$ since $X_n = J(X_n, D) \le J(P, D) \le P$ and $\mathcal{R}$ is an interval. If $P \notin \mathcal{R}$, then $P \in \mathcal{Q}_{n-1}$, so $n \ge 1$, and by definition of $\mathcal{U}_1$ and $\mathcal{U}_2$, either $d(P) > d(X_n)$, or $d(P) = d(X_n)$ and $J(P) > X_n$. If $d(P) > d(X_n)$, then $d(J(P, D)) = d(P) > d(X_n)$ since $J(P) \le J(P, D) \le P$, and $J(P, D) \notin \mathcal{R}$ since $d(R) = d(X_n)$ for all $R \in \mathcal{R}$. If $J(P) > X_n$, then $J(P) \notin \mathcal{R}$ since $J(R) = X_n$ for all $R \in \mathcal{R}$, and hence $J(P, D) \notin \mathcal{R}$ since $J(P, D) \ge J(P)$ and $\mathcal{R}$ is an interval. This proves (6).

Thus by Proposition 3.3, $L^k(\mathcal{E}; \mathcal{R}) = 0$ for all $k \ge k(p)$. Together with (5), this finishes the proof of (3), and hence of the theorem. □

4. Proof of Proposition 3.2

It remains to prove Proposition 3.2, which we restate here as:

Proposition 4.1. *Let (Γ, S, Y) be a reduced setup, set $D = Z(Y)$, and assume $\Gamma/C_\Gamma(D)$ is generated by quadratic best offenders on D. Set $\mathcal{F} = \mathcal{F}_S(\Gamma)$, and let $\mathcal{R} \subseteq \mathcal{F}^c$ be the set of all $R \ge Y$ such that $J(R, D) = Y$. Then $L^{k(p)}(\mathcal{F}; \mathcal{R}) = 0$.*

It is in this section that we use the classification of offenders by Meierfrankenfeld and Stellmacher [MS], and through that the classification of finite simple groups. Any proof of Proposition 4.4 without using these results would imply a classification-free proof of Proposition 4.1, and hence of Theorem 3.4.

In the following definition, when $H_1 \le H_2 \le \cdots \le H_k$ are subgroups of a group G, then $N_G(H_1, \ldots, H_k)$ denotes the intersection of their normalizers.

Definition 4.2. Let G be a finite group.

(a) A *radical p-subgroup* of G is a p-subgroup $P \le G$ such that $O_p(N_G(P)) = P$; i.e., $O_p(N_G(P)/P) = 1$.

(b) A *radical p-chain of length k* in G is a sequence of p-subgroups $P_0 < P_1 < \cdots < P_k \le G$ such that P_0 is radical in G, P_i is radical in $N_G(P_0, \ldots, P_{i-1})$ for each $i \ge 1$, and $P_k \in \mathrm{Syl}_p(N_G(P_0, \ldots, P_{k-1}))$.

The reason for defining this here is the following vanishing result, which involves only radical p-chains with $P_0 = 1$.

Proposition 4.3 ([AKO, Lemma III.5.27] and [O2, Proposition 3.5])**.** *Fix a finite group G, a finite $\mathbb{F}_p[G]$-module M, and $k \geq 1$ such that $\Lambda^k(G;M) \neq 0$. Then there is a radical p-chain $1 = P_0 < P_1 < \cdots < P_k$ of length k such that M contains a copy of the free module $\mathbb{F}_p[P_k]$.*

Other results similar to Proposition 4.3 were shown by Grodal, and follow from results in [Gr, § 5].

Note that since the trivial subgroup is a radical p-subgroup of G only if $O_p(G) = 1$, Proposition 4.3 includes the statement that $\Lambda^k(G;M) = 0$ if $O_p(G) \neq 1$ (Proposition 1.10(c)). The reason for defining radical p-chains more generally here — to also allow chains where $P_0 \neq 1$ — will be seen in the statement of the following proposition and in the proof of Proposition 4.1.

Proposition 4.4. *Let G be a nontrivial finite group with $O_p(G) = 1$, and let V be a faithful $\mathbb{F}_p[G]$-module. Fix a set $\mathcal{U}$ of nontrivial quadratic best offenders in G on V which is invariant under G-conjugacy, and assume $G = \langle \mathcal{U} \rangle$. Set $G_0 = O^p(G)$ and $W = C_V(G_0)[G_0, V]/C_V(G_0)$. Set $k = k(p)$, and let $P_0 < P_1 < \cdots < P_k$ be a radical p-chain in G such that P_0 does not contain any subgroup in $\mathcal{U}$. Then $C_W(P_0)$, with its induced action of P_k/P_0, does not contain a copy of the free module $\mathbb{F}_p[P_k/P_0]$.*

Proof. Quadratic offenders with faithful action are elementary abelian by Lemma 2.4. So the results in [MS] can be applied. In particular, $G = M = J_M(V) = J$ in the notation of [MS], $O^p(G) = \langle \mathcal{J} \rangle$ by [MS, Theorem 1(c)], and hence W as defined here is the same as W defined in [MS].

By [MS, Theorem 1(d,e)], W is a semisimple $\mathbb{F}_p[G]$-module, and each $U \in \mathcal{U}$ is a quadratic best offender on W. Set $W = W_1 \oplus \cdots \oplus W_m$, where each W_i is $\mathbb{F}_p[G]$-irreducible. Set $K_i = C_G(W_i)$.

Assume $C_W(P_0)$ contains a copy of $\mathbb{F}_p[P_k/P_0]$. Then by Lemma A.1(a), there is $1 \leq j \leq m$ such that $C_{W_j}(P_0)$ also contains a copy of $\mathbb{F}_p[P_k/P_0]$.

Choose $S \in \mathrm{Syl}_p(G)$ such that $P_k \leq S$, and set $T = S \cap K_j \in \mathrm{Syl}_p(K_j)$. Set $T_0 = T \cap P_0$ and $T^* = N_T(P_k)$. We claim that $T_0 = T^* = T$. Since W_j contains $\mathbb{F}_p[P_k/P_0]$, $C_{P_0}(W_j) = C_{P_k}(W_j)$, and hence $T_0 = P_0 \cap T = P_k \cap T$. If $T_0 < T$, then $P_k \ngeq T$, so $N_{P_kT}(P_k) > P_k$ by Lemma 1.13(a), and $T^* = N_T(P_k) > T_0$. For each $0 \leq i \leq k$,

$$[P_i, T^*] \leq [P_k, T^*] \leq P_k \cap T = T_0 \leq P_0 \leq P_i,$$

so $P_i \trianglelefteq T^*$. Thus $P_k < P_kT^* \leq N_G(P_0, \ldots, P_{k-1})$, contradicting the assumption that $P_k \in \mathrm{Syl}_p(N_G(P_0, \ldots, P_{k-1}))$. We conclude that $T = T_0 \leq P_k$.

Since $G = \langle \mathcal{U} \rangle$ is generated by quadratic best offenders on W, G/K_j is generated by quadratic best offenders on W_j by Lemma 2.2(a). So by Lemma 4.5, $P_kK_j/K_j \in \mathrm{Syl}_p(G/K_j)$. Thus $P_k = S \in \mathrm{Syl}_p(G)$, since $P_k \geq T \in \mathrm{Syl}_p(K_j)$.

Since $P_k \in \mathrm{Syl}_p(G)$ and $\mathcal{U}$ is G-conjugacy invariant, there is $U \in \mathcal{U}$ such that $U \leq P_k$. By assumption, $U \nleq P_0$. Set $U_0 = U \cap P_0$, and set $p^r = |U/U_0| > 1$. Since $C_W(U_0) \geq C_W(P_0)$ contains a copy of the free module $\mathbb{F}_p[U/U_0]$ of rank p^r, and since $|U||C_W(U)| \geq |U_0||C_W(U_0)|$,

$$p^r - 1 \leq \mathrm{rk}\big(C_W(U_0)/C_W(U)\big) \leq \log_p |U/U_0| = r.$$

Hence $p^r = |U/U_0| = 2$; i.e., $p = 2$ and $r = 1$. But $k = k(p) = 2$ since $p = 2$, so $|P_k/P_0| \geq 4$, and $C_W(U_0)$ contains at least two copies of the free module $\mathbb{F}_2[U/U_0]$

since it contains a copy of $\mathbb{F}_2[P_k/P_0]$. This implies $2 \le \operatorname{rk}\bigl(C_W(U)/C_W(U_0)\bigr) \le r = 1$. Since this is impossible, we conclude that $C_W(P_0)$ does not contain a copy of $\mathbb{F}_p[P_k/P_0]$. $\square$

The following lemma was needed to prove Proposition 4.4.

Lemma 4.5. *Let G be a nontrivial finite group, let W be a faithful, irreducible $\mathbb{F}_p[G]$-module, and assume G is generated by its quadratic best offenders on W. Let $P_0 < P_1 < \cdots < P_k$ be a radical p-chain in G with $k \ge k(p)$. Then either*

(a) *$C_W(P_0)$, with its induced action of P_k/P_0, does not contain a copy of the free module $\mathbb{F}_p[P_k/P_0]$; or*

(b) *$P_k \in \operatorname{Syl}_p(G)$.*

Proof. Quadratic offenders with faithful action are elementary abelian by Lemma 2.4. Also, $O_p(G) = 1$, since $C_W(O_p(G))$ is a nontrivial $\mathbb{F}_p[G]$-submodule of W and G acts faithfully. So the results in [MS] can be applied. Since the set $\mathcal{D}$ defined in [MS, Theorem 2] contains all quadratic offenders, $G = \langle \mathcal{D} \rangle$.

By [MS, Theorem 2], either G is a "genuine" group of Lie type in characteristic p; or $p = 2$ and G is one of the groups $3A_6$, A_7, Σ_n or A_n with the natural $\mathbb{F}_2[G]$-module, or $SO^{\pm}_{2m}(2^a)$ with the natural module. We consider these cases individually.

Case 1: Assume G is a genuine group of Lie type in characteristic p; in particular, G is quasisimple. The nontrivial radical p-subgroups of G are well known: by a theorem of Borel and Tits (see [BW, corollary] or [GL, 13-5]), they are all conjugate to maximal normal unipotent subgroups in parabolic subgroups. Hence the successive normalizers all contain Sylow p-subgroups of G, and the quotients are again groups of Lie type. Since $P_k \in \operatorname{Syl}_p(N_G(P_0, \dots, P_{k-1}))$, $P_k \in \operatorname{Syl}_p(G)$ in this case.

Case 2: Assume $p = 2$, and $G \cong 3A_6$ or A_7. Then the Sylow 2-subgroups of G have order 8, the nontrivial radical 2-subgroups have order 4 or 8, hence are normal in Sylow 2-subgroups, and thus $P_k \in \operatorname{Syl}_2(G)$ ($k \ge 2$).

Case 3: Assume $p = 2$, $G \cong \Sigma_m$ or A_m, and W is a natural module for G. Thus $\operatorname{rk}(W) = m - 1$ if m is odd, while $\operatorname{rk}(W) = m - 2$ if m is even.

Set $\boldsymbol{m} = \{1, 2, \dots, m\}$, with the action of G. Fix a radical p-chain $P_0 < P_1 < \cdots < P_k$ with $k \ge 2$, and assume $C_W(P_0)$ does contain a free submodule $\mathbb{F}_2[P_k/P_0]$. Thus $\operatorname{rk}(C_W(P_0)) \ge |P_k/P_0| \ge 4$.

If P_0 acts on $\boldsymbol{m}$ with more than one orbit, then by Lemma A.4, $|\boldsymbol{m}/P_0| \ge 3$, and $C_W(P_0)$ is contained in $\mathbb{F}_2(\boldsymbol{m}/P_0)/\Delta$ where $\Delta \cong \mathbb{F}_2$ is generated by the sum of the basis elements. Then $\mathbb{F}_2(\boldsymbol{m}/P_0)$ also contains a copy of the free module $\mathbb{F}_2[P_k/P_0]$, and hence $\boldsymbol{m}/P_0$ contains a free (P_k/P_0)-orbit by Lemma A.1(b). But this is impossible by Lemma A.3.

Now assume P_0 acts transitively on $\boldsymbol{m}$. We have $P_0 = \overline{P}_0 \cap G$, where $\overline{P}_0 = O_2(N_{\Sigma_m}(P_0))$ is radical in Σ_m. By [AF, p. 7], $\overline{P}_0 = E_{s_1} \wr E_{s_2} \wr \cdots \wr E_{s_n} \le \Sigma_m$ ($n \ge 1$), where $E_s \cong C_2^s$ with a free action on a set of order 2^s, and where P_0 acts on $\boldsymbol{m}$ via the canonical action of a wreath product ($m = 2^{s_1 + \ldots + s_n}$). Let $Q \le P_0$ be the stabilizer of a point under the action on $\boldsymbol{m}$. Then $P_0/\langle \operatorname{Fr}(P_0), Q \rangle \cong E_{s_n}$, so $C_W(P_0) \cong \mathbb{F}_2^{s_n}$ by Lemma A.4. Also, $m \ge 8$ since $4 \bigl| |P_k/P_0| \bigr| |N_G(P_0)/P_0|$, so either $\overline{P}_0 \not\le A_m$ (if $s_1 = 1$), or $N_{\Sigma_m}(P_0) \le A_m$ (if $s_1 \ge 3$, or $s_1 = 2$ and $n \ge 2$). In both

cases, $N_G(P_0)/P_0 \cong N_{\Sigma_m}(\overline{P}_0)/\overline{P}_0 \cong \prod_{i=1}^n GL_{s_i}(2)$ ([AF, p. 6]), and acts on $C_W(P_0)$ via projection to the factor $GL_{s_n}(\mathbb{F}_2)$. As seen in Case 1, any radical 2-chain in $GL_{s_n}(\mathbb{F}_2)$ must end with a Sylow 2-subgroup; i.e., $P_k/P_0 \cong U \in \mathrm{Syl}_2(GL_{s_n}(\mathbb{F}_2))$. But the condition $|P_k/P_0| = 2^{s_n(s_n-1)/2} \le \mathrm{rk}(C_W(P_0)) = s_n$ then implies $s_n \le 2$, which is impossible since $|P_k/P_0| \ge 4$.

Case 4: Now assume $p = 2$ and $G \cong SO^{\pm}_{2m}(q)$, where $q = 2^a$ ($a \ge 1$), and W is the natural $\mathbb{F}_2[G]$-module of rank $2am$. Then $G \not\cong SO^+_4(2) \cong GL_2(2) \wr C_2$, since G is not generated by best offenders in this case (W splits in a unique way $W = W_1 \oplus W_2$ where $W_1 \perp W_2$ and $\mathrm{rk}(W_i) = 2$, and all best offenders send each W_i to itself). Also, $G \not\cong SO^-_4(2) \cong \Sigma_5$ (with the reduced permutation action on W) since this was already eliminated in Case 3. Thus either $m \ge 3$, or $m = 2$ and $q \ge 4$ ($a \ge 2$). Set $G_0 = \Omega^{\pm}_{2m}(q)$, so $[G{:}G_0] = 2$.

For any nontrivial radical 2-subgroup $1 \ne P \le G$, $P \cap G_0$ is a radical p-subgroup of G_0, and hence is either trivial, or is a maximal normal unipotent subgroup in a parabolic subgroup. If $P \cap G_0 = 1$ (and $P \ne 1$), then $P = \langle t \rangle$ for some involution $t \in SO^{\pm}_{2m}(q) \smallsetminus \Omega^{\pm}_{2m}(q)$. By [AS, 8.10], t has type b_ℓ (in the notation of [AS, § 7–8]), where $\ell = \mathrm{rk}([t, W])$ is odd. By [AS, 8.7], $O_2(N_G(t))$ contains all transvections $u \in G$ with $[u, W] \le [t, W]$, and hence $P = \langle t \rangle$ is radical only if t is itself a transvection ($\ell = 1$). In that case, by [AS, 8.7] again, $N_G(t)/\langle t \rangle \cong Sp_{2m-2}(q)$.

Assume first that $P_0 = 1$. If $P_1 \cap G_0 = 1$, then P_1 is generated by a transvection, so $\mathrm{rk}([P_1, W]) = 2$, and W does not contain a free $\mathbb{F}_2[P_2]$-module for any $P_2 > P_1$. Thus $P_1 \cap G_0$ is a maximal normal unipotent subgroup of a parabolic subgroup. So $|P_1| \ge q^{2m-3}$ (Lemma A.5), $|P_2| \ge q^{2m-2}$, and $q^{2m-2} > \mathrm{rk}_{\mathbb{F}_2}(W) = 2am$ since we assume $(q, m) \ne (2, 2)$. Hence this case is impossible.

Next assume $P_0 \ne 1$ and $P_0 \cap G_0 = 1$. As noted above, P_0 is generated by a transvection (hence $\mathrm{rk}_{\mathbb{F}_q}(C_W(P_0)) = 2m - 1$), and $N_G(P_0)/P_0 \cong Sp_{2m-2}(q)$. By the argument used in Case 1, $P_k/P_0 \in \mathrm{Syl}_2(N_G(P_0)/P_0)$, and thus has order $q^{(m-1)^2}$. Thus $|P_k/P_0| > \mathrm{rk}_{\mathbb{F}_2}(C_W(P_0)) = a(2m - 1)$ if $m \ge 3$. If $m = 2$, then $N_G(P_0)/P_0 \cong Sp_2(q)$, all nontrivial radical 2-subgroups are Sylow subgroups, and so there is no radical 2-chain of length 2.

Finally, assume $P_0 \cap G_0 \ne 1$, and hence is a maximal normal unipotent subgroup of a parabolic subgroup. Then $C_W(P_0) \le C_W(P_0 \cap G_0)$ is a totally isotropic subspace $W_0 < W$ of rank $\ell \le m$, and $N_G(P_0)/P_0 \cong GL(W_0) \times N_{SO(W_0^\perp/W_0)}(P_0)/P_0$ acts on it via projection to the first factor. So P_k/P_0 is contained in the factor $GL(W_0)$ since it acts faithfully on $C_W(P_0)$, $\ell \ge 3$ since otherwise there is no radical 2-chain of length ≥ 2, and P_k/P_0 is a Sylow subgroup by the usual argument. Thus $|P_k/P_0| = q^{\ell(\ell-1)/2}$, and $|P_k/P_0| > a\ell = \mathrm{rk}_{\mathbb{F}_2}(C_W(P_0))$ since $\ell \ge 3$. □

Proof of Proposition 4.1. Fix a reduced setup (Γ, S, Y), set $D = Z(Y)$, $V = \Omega_1(D)$, and $G = \Gamma/C_\Gamma(D)$, and assume $G = \langle \mathcal{U} \rangle$ where

$$\mathcal{U} = \{1 \ne P \le G \mid P \text{ a quadratic best offender on } D\}\ .$$

We assume inductively that Proposition 4.1, and hence also Proposition 3.3, hold for all (Γ^*, S^*, Y^*) with $|\Gamma^*| < |\Gamma|$. Note that $\mathcal{U}$ is a set of quadratic best offenders on V by Lemma 2.2(a), and $O_p(G) = 1$ by definition of a reduced setup. Hence G acts faithfully on V, since $C_G(V)$ is a p-subgroup by Lemma 1.14 (so $C_G(V) \le O_p(G)$).

For each $H \leq \Gamma$, let $\overline{H} = HC_\Gamma(D)/C_\Gamma(D)$ be the image of H in G. Recall that $Y = C_S(D) \in \mathrm{Syl}_p(C_\Gamma(D))$ by definition of a reduced setup. So for each $P \in \mathscr{S}(S)_{\geq Y}$, $N_\Gamma(PC_\Gamma(D)) = N_\Gamma(P)C_\Gamma(D)$ since $P \in \mathrm{Syl}_p(PC_\Gamma(D))$. Thus

$$N_G(\overline{P}) = \overline{N_\Gamma(P)} \quad \text{whenever} \quad Y \leq P \leq S. \tag{1}$$

Recall that $\mathcal{R} = \{P \in \mathcal{F}^c \mid J(P,D) = Y\}$. Set $\mathcal{Q} = \mathscr{S}(S)_{\geq Y} \smallsetminus \mathcal{R}$. Thus $\mathcal{Q}$ is the set of all $P \in \mathscr{S}(S)_{\geq Y}$ such that $\overline{P} \cong P/Y$ contains at least one nontrivial best offender on D, which can be assumed quadratic by Timmesfeld's replacement theorem (Theorem 2.5); and $\mathcal{R}$ is the set of all P such that $\overline{P}$ contains no such best offender.

Set $D_0 = 1$. For each $i \geq 1$, set $D_i = \Omega_i(D) = \{g \in D \mid g^{p^i} = 1\}$ and $V_i = D_i/D_{i-1}$. Thus each V_i is an $\mathbb{F}_p[G]$-module, and $(x \mapsto x^p)$ sends V_i injectively to V_{i-1} for each $i > 0$.

Set $k = k(p)$. We will show that $\Lambda^k(\mathrm{Out}_\Gamma(R); Z(R)) = 0$ for each $R \in \mathcal{R}$; the proposition then follows from Corollary 1.9. Here, $Z(R) = C_D(R)$ and $\mathrm{Out}_\Gamma(R) \cong N_\Gamma(R)/R$: since $R \geq Y$ and $C_S(Y) = Z(Y) = D$. So by Proposition 1.10(d), it suffices to show, for each R and i, that $\Lambda^k\big(N_\Gamma(R)/R; C_{D_i}(R)/C_{D_{i-1}}(R)\big) = 0$. Also, for each i, $C_{D_i}(R)/C_{D_{i-1}}(R)$ can be identified with an $N_\Gamma(R)$-invariant subgroup of $C_{V_i}(R) \leq C_V(R)$. It thus suffices to show that

$$\Lambda^k\big(N_\Gamma(R)/R; X\big) = 0 \quad \forall\, R \in \mathcal{R},\, \forall\, N_\Gamma(R)\text{-invariant } X \leq C_V(R).$$

Set $V_1 = C_{O^p(G)}(V)$, $V_2 = V_1[O^p(G), V]$, and $W = V_2/V_1$. Thus the G-actions on V_1 and on V/V_2 factor through the p-group $G/O^p(G)$. So by Proposition 1.10(a,b,c), for each $R \in \mathcal{R}$, and each $N_\Gamma(R)$-invariant $X \leq V_1$ and $Y \leq V/V_2$, $\Lambda^k(N_\Gamma(R)/R; X) = 0$ and $\Lambda^k(N_\Gamma(R)/R; Y) = 0$. By the exact sequences of Proposition 1.10(d), we are now reduced to showing

$$\Lambda^k\big(N_\Gamma(R)/R; X\big) = 0 \quad \forall\, R \in \mathcal{R},\, \forall\, N_\Gamma(R)\text{-invariant } X \leq C_W(R).$$

Assume otherwise. By Proposition 4.3, there is a radical p-chain $R = P_0 < P_1 < P_2 < \cdots < P_k$ in Γ such that $C_W(P_0)$ contains a copy of the free module $\mathbb{F}_p[P_k/P_0]$. By (1), $\overline{R} = \overline{P}_0 < \overline{P}_1 < \overline{P}_2 < \cdots < \overline{P}_k$ is a radical p-chain in G. But this contradicts Proposition 4.4, since by assumption, $R \in \mathcal{R}$ implies that $\overline{R}$ does not contain any subgroups in $\mathcal{U}$. $\square$

Appendix A. Radical p-chains and free submodules

We collect here some lemmas needed in the proofs in Section 4.

Lemma A.1. *Let P be a p-group, and let V be an $\mathbb{F}_p[P]$-module which contains a copy of the free module $\mathbb{F}_p[P]$.*

(a) *If $V = V_1 \oplus \cdots \oplus V_n$ where the V_i are $\mathbb{F}_p[P]$-submodules, then for some $i = 1, \ldots, n$, V_i contains a copy of $\mathbb{F}_p[P]$.*

(b) *If V contains an $\mathbb{F}_p$-basis $\boldsymbol{b}$ which is permuted by P, then $\boldsymbol{b}$ contains a free P-orbit.*

Proof. **(a)** Set $F=\mathbb{F}_p[P]$ for short. Let $f\colon F \longrightarrow V$ be a P-linear monomorphism. For each i, let $\mathrm{pr}_i\colon V\to V_i$ be the projection, and set $f_i=\mathrm{pr}_i\circ f$. Choose i such that f_i sends $C_F(P)\cong\mathbb{F}_p$ injectively to V_i. Then f_i is injective, since otherwise $\mathrm{Ker}(f_i)\neq 0$ would have trivial fixed subspace. Thus V_i contains a copy of F.

(b) Write $V=V_1\oplus\cdots\oplus V_n$, where each V_i has as basis one P-orbit $\boldsymbol{b}_i\subseteq\boldsymbol{b}$. By (a), there is i such that V_i contains a copy of $\mathbb{F}_p[P]$, and then $\boldsymbol{b}_i$ is a free orbit. □

The next three lemmas involve radical 2-subgroups and radical 2-chains in symmetric and alternating groups.

Lemma A.2. *Let $G=\Sigma_m$ or A_m, with its canonical action on $\boldsymbol{m}=\{1,\dots,m\}$. Let $P_0<P_1<\cdots<P_k$ be a radical p-chain in G. Fix $0\leq j\leq k$, let $X_1,\dots,X_r$ be the P_j-orbits on $\boldsymbol{m}$, let $\overline{H}\leq\Sigma_m$ be the subgroup of those permuations which send each X_i to itself, and set $H=\overline{H}\cap G$. Thus $\overline{H}=H_1\times\cdots\times H_r$, where H_i is the symmetric group on X_i. Let $Q_{ji}\leq H_i$ be the image of $P_j\leq H$ under the i-th projection, and set $\overline{Q}_j=Q_{j1}\times\cdots\times Q_{jr}$ and $Q_j=\overline{Q}_j\cap G$. Then $Q_j=P_j$.*

Proof. For each $0\leq\ell\leq j$, let $Q_{\ell i}\leq H_i$ be the image of P_ℓ under the i-th projection, and set $\overline{Q}_\ell=Q_{\ell 1}\times\cdots\times Q_{\ell r}$ and $Q_\ell=\overline{Q}_\ell\cap G$. We can assume by induction on j that $Q_\ell=P_\ell$ for each $\ell<j$. Also, $Q_{\ell i}\trianglelefteq Q_{ji}$ for each i since $P_\ell\trianglelefteq P_j$, and so $P_\ell=Q_\ell\trianglelefteq Q_j$.

Set $G_j=N_G(P_0,\dots,P_{j-1})$ for short. We just showed that $Q_j\leq G_j$. Each element of Σ_m which normalizes P_j must permute the X_i and hence also normalizes Q_j. Thus $N_{Q_j}(P_j)\trianglelefteq N_{G_j}(P_j)$, so $N_{Q_j}(P_j)=P_j$ since P_j is radical in G_j, and $Q_j=P_j$ by Lemma 1.13(a). □

Lemma A.3. *Let $G=\Sigma_m$ or A_m, with its canonical action on $\boldsymbol{m}=\{1,\dots,m\}$. Let $P_0<P_1<\cdots<P_k$ be a radical p-chain in G of length $k\geq 2$. Then the action of P_k/P_0 on $\boldsymbol{m}/P_0$ has no free orbit.*

Proof. We use the notation of Lemma A.2 and its proof, with $j=1$. Note that $\overline{Q}_0$ acts trivially on $\boldsymbol{m}/P_0$ since each factor Q_{0i} acts trivially on X_i/P_0, so $\overline{Q}_1/\overline{Q}_0$ acts on $\boldsymbol{m}/P_0$. We regard P_1/P_0 as a subgroup of $\overline{Q}_1/\overline{Q}_0$. Since Q_{1i} acts transitively on X_i for each i, Q_{1i}/Q_{0i} acts transitively on X_i/P_0.

Assume there is a free P_k/P_0-orbit on $\boldsymbol{m}/P_0$, and set $s=|P_k/P_1|$. After reindexing the X_i if necessary, we can assume P_1/P_0 acts freely on each of its orbits X_i/P_0 for $1\leq i\leq s$, and that P_k/P_1 permutes $\{X_1,\dots,X_s\}$ transitively. Also, P_1/P_0 has index at most two in $\overline{Q}_1/\overline{Q}_0$, where this group splits as a product of subgroups Q_{1i}/Q_{0i}, each acting on an orbit X_i/P_0. This is possible only if $s=|P_k/P_1|=2$, $|P_1/P_0|=2$, $|\overline{Q}_1/\overline{Q}_0|=4$, and P_k/P_1 exchanges X_1 and X_2 (so $|X_1|=|X_2|$). Note that $\overline{Q}_0=P_0$ since $[\overline{Q}_1/\overline{Q}_0{:}P_1/P_0]=2$.

Fix $\sigma\in P_1\smallsetminus P_0$; thus σ acts on $\boldsymbol{m}/P_0$ as a product of two transpositions. For $i=1,2$, write $X_i=X_i'\amalg X_i''$, where X_i' and X_i'' are P_0-orbits, and where σ exchanges X_i' with X_i''. Write $\sigma=\sigma_1\cdots\sigma_r$ with $\sigma_i\in Q_{1i}$. If σ_i is an even permutation ($i=1,2$), then $(\sigma_i\times\mathrm{Id}_{\boldsymbol{m}\smallsetminus X_i})\in P_1$, which is impossible by the above description of P_1/P_0. Hence σ_1 and σ_2 are odd permutations.

Assume $|X_i'|=\frac{1}{2}|X_i|=2^t>1$ $(i=1,2)$. Write $\sigma_i=\sigma_i'\tau_i$, where $\tau_i^2=1$, τ_i exchanges X_i' and X_i'', and σ_i' permutes X_i' and is the identity on X_i''. Then σ_i' is

odd since τ_i is even (a product of 2^t transpositions), and $\sigma_i^2 = \sigma_i'\sigma_i'' \in Q_{0i}$ where $\sigma_i'' = \tau_i\sigma_i'\tau_i$ permutes X_i'' and is the identity on X_i'. Then $\sigma_1'\sigma_2' \in P_0$ is the product of an odd permutation on X_1 and one on X_2, which is impossible since $\overline{Q}_0 = P_0$.

Thus $|X_1'| = 1$, so $|X_1| = |X_2| = 2$, and σ acts on the fixed subset $C_{\boldsymbol{m}}(P_0)$ as a product of two transpositions. Hence $N_{\Sigma_m}(P_0, P_1)/P_1$ contains a direct factor $N_{\Sigma_4}(\bar{\sigma})/\langle\bar{\sigma}\rangle \cong C_2^2$ (where $\bar{\sigma} = (1\,2)(3\,4)$), so $N_G(P_0, P_1)/P_1$ contains the normal subgroup $N_{A_4}(\bar{\sigma})/\langle\bar{\sigma}\rangle \cong C_2$. But this contradicts the assumption that P_1 is radical in $N_G(P_0)$. □

Lemma A.4. *Let $G = \Sigma_m$ or A_m, with its canonical action on $\boldsymbol{m} = \{1, \ldots, m\}$. Set $V = \mathbb{F}_2(\boldsymbol{m})$, regarded as an $\mathbb{F}_2[G]$-module. Let $\Delta \le V$ be the 1-dimensional submodule generated by the sum of the elements in $\boldsymbol{m}$. Then for each radical 2-subgroup $P \le G$,*

- *$C_{V/\Delta}(P) = C_V(P)/\Delta$ if $|\boldsymbol{m}/P| \ge 3$;*
- *$\mathrm{rk}_{\mathbb{F}_2}(C_{V/\Delta}(P)) \le 2$ if $|\boldsymbol{m}/P| = 2$; and*
- *$C_{V/\Delta}(P) \cong \mathrm{Hom}(P/\langle \mathrm{Fr}(P), Q\rangle, \mathbb{F}_2)$ if P acts transitively on $\boldsymbol{m}$ with isotropy subgroup (point stabilizer) Q.*

Proof. The exact sequence $1 \to \Delta \to V \to V/\Delta \to 1$ induces the following exact sequence in cohomology:

$$1 \longrightarrow \Delta \longrightarrow C_V(P) \longrightarrow C_{V/\Delta}(P) \longrightarrow H^1(P;\Delta) \xrightarrow{\ \psi\ } H^1(P;V).$$

Here, $H^1(P;\Delta) \cong \mathrm{Hom}(P, \mathbb{F}_2)$. If $x_1, \ldots, x_r \in \boldsymbol{m}$ are orbit representatives for the P-action, and $Q_i \le P$ is the stabilizer subgroup at x_i, then

$$H^1(P;V) \cong \bigoplus_{i=1}^{r} H^1(P; \mathbb{F}_2(P/Q_i)) \cong \bigoplus_{i=1}^{r} \mathrm{Hom}(Q_i, \mathbb{F}_2).$$

Also, ψ is defined by restriction under these identifications, and so $\mathrm{Ker}(\psi) \cong \mathrm{Hom}(P/Q, \mathbb{F}_2)$ where $Q = \langle \mathrm{Fr}(P), Q_1, \ldots, Q_r\rangle$. We thus get a short exact sequence

$$1 \longrightarrow C_V(P)/\Delta \longrightarrow C_{V/\Delta}(P) \longrightarrow \mathrm{Hom}(P/Q, \mathbb{F}_2) \longrightarrow 1,$$

where $\mathrm{rk}(C_V(P)) = |\boldsymbol{m}/P|$.

This proves the lemma when $r = |\boldsymbol{m}/P| = 1$. If $r > 1$, then by Lemma A.2 (applied with $j = 0$), $P = \overline{P} \cap G$ for some $\overline{P} = P_1 \times \cdots \times P_r \le \Sigma_m$, where each P_i permutes the P-orbit Px_i transitively. For each i, P_i is contained in the stabilizer of x_j for $j \ne i$, so $P_i \cap G \le Q$. Thus $[P{:}Q] \le 2$ if $r = 2$. If $r \ge 3$, then $P_iP_j \cap G \le Q$ for each $i, j \in \{1, \ldots, r\}$, and so $P = Q$ in this case. The remaining cases of the lemma now follow from the above exact sequence. □

We also needed lower bounds for orders of radical subgroups of $\Omega_{2m}^{\pm}(q)$.

Lemma A.5. *Let P be a radical 2-subgroup of $G = \Omega_{2m}^{\pm}(q)$, where $m \ge 2$ and $q = 2^a$. Then $|P| \ge q^{2m-3}$, and $|P| \ge q^{2m-2}$ if $m \ge 4$.*

Proof. By a theorem of Borel and Tits (see [BW, corollary] or [GL, 13-5]), P is conjugate to the maximal normal unipotent subgroup in a parabolic subgroup $N < G$. Thus $P \cong O_2(N)$, N contains a Borel subgroup and hence a Sylow 2-subgroup of G, and $N/O_2(N)$ is a semisimple group of Lie type with Dynkin diagram strictly contained in that of $G = D_n(q)$.

Thus G has Sylow 2-subgroups of order $q^{m(m-1)}$, while $N/O_2(N)$ must be contained in D_{m-1} (Sylow of order $q^{(m-1)(m-2)}$), A_{m-1} (order $q^{m(m-1)/2}$), or $A_\ell \times D_{m-\ell-1}$ (smaller order). Hence $|P| \geq \min\{q^{2(m-1)}, q^{m(m-1)/2}\} \geq q^{2m-3}$. When $G = \Omega_{2m}^-(q)$, the argument is the same, except that we only need consider subgroups whose Dynkin diagram is invariant under the graph automorphism for D_m. $\square$

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LAGA, Institut Galilée, Av. J-B Clément, 93430 Villetaneuse, France

E-mail address: bobol@math.univ-paris13.fr